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"A mathematical model for linear and non-linear penetrative convection due to a S-shaped temperature profile with heat flux prescribed on the boundaries"

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Abstract

In this paper, a mathematical model to study linear and nonlinear penetrative convection in anice-water medium subject to an S-shaped cubic temperature profile is presented and discussed.The boundaries are such that they heat flux prescribed on them.The analysis is extended to study the behaviour of nonlinear disturbances for Rayleigh numbers near the critical value. It is of interest to observe whether finite amplitude motion can exist for Rayleigh numbers less than the critical value. By applying the modified power series technique, solvability condition and the evolution equations are derived. This problem includes cubic temperature profile in an ice-water medium. The present study provides a better understanding of a realistic geothermal model. The methodology associated with the study is of special interest and has some novel features.

Introduction:

In recent years, the study of the phenomenon of the convective process in a horizontal fluid/porous layer has received remarkable attention owing to its very wide applications in Science, Engineering and Industrial areas. Convection can be the dominant mode of heat and mass transport in many processes that involve the freezing and melting of the storage material. Natural convection is an omnipresent transport phenomenon in saturated porous geological structures. In fact, the study of convection is of most importance in geophysical, astrophysical and heat transfer problems. For example, the extraction of energy from geothermal sources is the most promising one among the other methods and it is believed that the fluid in these reservoirs are highly permeable and consists of multi-components rather than a single component. Therefore, buoyancy driven convection in a porous medium with water as the working fluid is an important mechanism of energy transport. In fact, the key feature of a major geothermal system while on the land or beneath the sea floor is a high intrinsic heat transport. In fact, the local thermal conditions and the physical properties of media are directly of great importance on the characteristics of the heat and mass transfer in such real configurations. Moreover, the nature of the fluid flow (2D, 3D, pattern and range) is drastically dependent on the complexity of geophysical sites, i.e., the geometry, heterogeneity and anisotropy of the domains. Fluid motions induced by free convection have tangible effects in geothermal areas, on the diffusion of pollutants or on the mineral diagenesis processes. The multi-component onset of convection is important in many naturally occurring phenomena and technological processes. In double diffusive convection, heat and solute diffuse at different rates, as a result of which complex flow structures may form which have no counterpart in buoyant flows driven by a single component. Extensive literature pertaining to this phenomenon is available (Fujita and Gosting (1956); Miller (1966); Miller et al. (1967); Nield (1968); Hurle and Jakeman (197 1); Huppert and Manins (1972); Schechter et al (1972); Velarde and Schechter (1972); Vitaliano et al. (1972); Caldwell (1974); Turner (1979,1985); Griffiths (1979); Antoranz and Vegiar (1979); Leaist and Lyons (1980); Placsek and Toomre (1980); Narusawa and Suzukawa (1981); Srimani (1981); Takao, Tsuchiya and Narasuwa (1982); McTaggart (1983); Srimani (1984, 1991); Torrones and Pearistein (1989); Anamika (1990); Chen and Su (1992); Zimmermann Muller and Davis (1992); Tanny, Chen and Chen (1994); Shivakumar (1997); Skarda, Jacqmin and McCaughan (1998);). In this review, an additional effect viz., the effect of the coupled fluxes of the two properties due to irreversible thermodynamic effects is considered. This is termed as the effect of Coupled diffusion or Cross-diffusion and Sort effect is an example of this cross-diffusion where a flux of salt is caused by a spatial gradient of temperature. In fact, Dufour effect is the corresponding flux of heat caused by a spatial gradient of temperature.

Some literature (Hurle and Jakeman (1971); Skarda, Jacqmin and McCaughan (1998); Zimmermann et al (1992); Chen and Chen (1994)) in this direction are useful. McDougall (1983) has made a detailed study of Double-diffusive convection caused by coupled molecular diffusion. The results of his linear stability analysis, predicts that for a sufficiently large coupled diffusion effect, fingers can form even when the concentrations of both the components make the fluid's density gradient statically stable. The conditions for the occurrence of finger as well diffusive instabilities are predicted. The results of his finite amplitude analysis reveal that for sufficiently large cross-diffusion effect, fingers do exist.

The present paper aims at answering the following questions that arise in the study:

i) What are the critical conditions for the onset of penetrative convection under the prescribed conditions?

ii) What is the influence of the S-shaped cubic temperature profile on the zeroth-order solution?

iii)Arethere any marked differences ?

iv) Is it possible to predict when a gravitationally unstable two-component layerlies above a thinner stable layer?

v) What is the nature of bifurcation?

vi) Do local minima exist in the Rayleigh number curve? Are there any marked differences between an ordinary penetrative convection and a penetrative porous convection?

vii) What is the nature of the density dependence on temperature and solute concentration?

viii) What is the cumulative effect of the governing parameters on the wave, temperature and concentration profiles?

ix) What is the nature of the differential equation associated with the horizontal and temporal structure of double diffusive penetrative convection?

x) What is the nature of penetration under different physically feasible conditions?

xi)Is it possible to suppress or enhance penetrative double diffusive convection by considering a suitable choice of the parameters?

2. Formulation and solution of the problem

The physical configuration is as shown in figure 1, where a sparsely packed ice-water layer of infinite horizontal extent is confined between two plates which have fixed heat flux boundary condition defined on them. No-slip velocity boundary conditions are imposed on the two boundaries.

The idealized equation of state for the fluid will be taken to be

$$\rho = \rho_0[1 - \alpha(T - T^*)^2]$$

where ρ_0, α and T^* are absolute constants of the fluid, in water the values $\rho_0 = 1\text{g/cm}^3$, $\alpha = 8 \times 10^{-6} K^{-2}$ and $T^* = 3.98^0$C give a reasonably accurate prescription. Also let,

$$T = T^* + [A(z-d)^3 - \beta(z-d)] + \theta(x,y,z,t)$$

Where$T^* + [A(z-d)^3 - \beta(z-d)] + \theta(x,y,z,t)$ is the static temperature distribution,θ is the deviations from the static state due to the convective motion. For ice-covered lakes A

and βarea both negative. Here $z = d$ corresponds to the vertical position of the static density maximum and correspondingly, for $0 < z < d$ the fluid under consideration is gravitationally unstable, while for $d < z <$ □ it is stably stratified. Using the above forms for the density dependence and the background temperature field, non-dimensionalizing quantities with respect to the length$(\beta/A)^{\frac{1}{2}}d$, the velocity $\kappa/(\beta/A)^{\frac{1}{2}}$, the time d^2/κ and the temperature βd. The set of two-dimensional, dimensionless Boussinesq equations is

$$\frac{\partial \theta}{\partial t} + (3z^2 - 6z + 2)\frac{\partial \psi}{\partial x} + \frac{\partial(\psi,\theta)}{\partial(x,z)} = \nabla^2\theta \qquad (1)$$

$$Pr^{-1}\left[\nabla^2\psi_t + \frac{\partial(\psi,\nabla^2\psi)}{\partial(x,z)}\right] = R[(z-1)^3 - (z-1+\theta)]\theta_x + \nabla^4\psi \qquad (2)$$

where $Pr = \nu/\kappa$ is the Prandtl number,
$R = 2\alpha g\beta^2 d^5/\kappa\nu$ is the Penetration Rayleigh number.
The two boundaries are assumed to be rigid and hence the velocity and temperature boundary conditions in the dimensionless form are given by
$\psi = \psi_z = 0$ on $z = 0$ and □
$\theta_z = 0$ on $z = 0$ and □
where $q = (-\psi_z, 0, \psi_x)$ is utilized.

Linear stability analysis

For all values of the parameters R, □ and Pr there exists the exact solution
$\theta = \psi = 0$ for all x, z, t
to the set of equations (1) and (2). This solution is the basic state solution corresponds to the linear temperature variation. The aim of section is to describe the critical values of the parameters at which the system becomes unstable to small perturbations. Therefore, we assume the solution of the linearized equations to be of the form

$$\left.\begin{aligned} \theta &= \mathrm{Re}\{\theta(z)\exp[st + iax]\} \\ \psi &= \mathrm{Re}\{ia\Psi(z)\exp[st + iax]\} \end{aligned}\right\}$$

so that the disturbance has an assumed horizontal wavenumber 'a' and is some temporal growth rate s. In general, s may be complex (i.e., $s = s_r + is_i$). The eigenvalue problem for $R_c(a,$ □$)$ is thus

$$\left.\begin{aligned} (D^2 - a^2 - s)\theta &= -(3z^2 - 6z + 2)a^2\Psi \\ (D^2 - a^2)\left(D^2 - a^2 - \frac{s}{Pr}\right)\Psi &= R[(1-z)^3 - (1-z)]\theta \\ D\theta = \Psi = D\Psi &= 0 \text{ on } z = 0 \text{ and } □ \end{aligned}\right\} \qquad (3)$$

where D stands for the operator d/dz. The condition for marginal stability in the case of ordinary convection is $\mathrm{Re}\{s\} = 0$and for the case of convection with constraints $\mathrm{Re}\{s\} = 0$ and $\mathrm{Im}\{s\} = 0$.

From (3), it is evident that in the penetrative convection with cubic temperature profile, we get a sixth order differential equation with six boundary conditions. A perturbation expansion about zero wavenumber is constructed for R_c, θ and Ψ as

$$\left.\begin{aligned} R_c &= R_{c0} + a^2R_{c2} + a^4R_{c4} + \cdots \ldots\ldots\ldots\ldots \\ \theta &= \theta_0(z) + a^2\theta_2(z) + a^4\theta_4(z) + \cdots \ldots\ldots \\ \Psi &= \Psi_0(z) + a^2\Psi_2(z) + a^4\Psi_4(z) + \cdots \ldots \end{aligned}\right\} \qquad (4)$$

Observe that the horizontal wave number occurs in powers of a^2, because of the inherent symmetry in the problem. The substitution of (5) into (4) yields a sequence of solvable equations when like powers of a^2 are equated. The zeroth-order solution is simply

$$\left.\begin{array}{l}\theta_0 = 1 \\ \Psi_0 = \frac{R_{c0}}{5!}\left[-\frac{z^7}{7} + z^6 - 2z^5 + \Box z^3\left(6\Box - 4\Box^2 + \frac{5\Box^3}{7}\right) + \Box^2 z^2\left(-4\Box + 3\Box^2 - \frac{4\Box^3}{7}\right)\right]\end{array}\right\} \quad (5)$$

From the solvability condition R_{c0} is determined and is simply

$$R_{c0} = \frac{6!}{\Box^4(-6\Box^2 + 6\Box - 2)\left[\left(\frac{5\Box^3}{28} - \frac{6\Box^2}{7} + \frac{3\Box}{2} - 1\right) + \left(1 - \frac{\Box}{2}\right)\right]}$$

It is to be noted that the vertical structure of the saturated fluid velocity is apparent from the expression for $\Psi_0(z)$.

Weakly nonlinear analysis

The analysis is restricted to relatively shallow layers where long horizontal scales arepreferred. We study the behavior of the system to nonlinear disturbances, near the critical Rayleigh number. It is assumed that the penetrative convection in the presence of S-shaped cubic profile occurs on a long horizontal scale. For this purposewe introduce the scale

$\xi = \varepsilon x, \quad \tau = \varepsilon^4 t$

where ε is a small parameter of the same magnitude as the horizontal wavenumber. Expand the variables in powers of ε^2 as

$$\left.\begin{array}{r}R = R_0 + \varepsilon^2 R_2 + \varepsilon^4 R_4 + \varepsilon^6 R_6 + \cdots \ldots\ldots\ldots\ldots\ldots\ldots \\ \theta(x,z,t) = \varepsilon^2\theta_0(\xi,z,\tau) + \varepsilon^4\theta_2(\xi,z,\tau) + \varepsilon^6\theta_4(\xi,z,\tau) + \cdots \ldots\ldots \\ \Psi(x,z,t) = \varepsilon^3\Psi_0(\xi,z,\tau) + \varepsilon^5\Psi_2(\xi,z,\tau) + \varepsilon^7\Psi_4(\xi,z,\tau) + \cdots \ldots\ldots\end{array}\right\} \quad (6)$$

Substituting the expansion (6) into the governing equations (3) and equating like powers of ε, we obtain the following set of equations

$$D^2\theta_n = (3z^2 - 6z + 2)\Psi_{n-2\xi} - \theta_{n-2\xi\xi} + \theta_{n-4\tau} + \sum_{m=4}^{n}\frac{\partial(\Psi_{m-4}, \theta_{n-m})}{\partial(\xi,z)} \quad (7)$$

$$D^4\Psi_n = -\sum_{m=0}^{n} R_{n-m}(z-1)^3\theta_{m\xi} + \sum_{m=0}^{n} R_{n-m}(z-1)\,\theta_{m\xi} - \sum_{m=2}^{n} R_{n-m}\left[\sum_{i=2}^{m}\theta_{m-i}\theta_{i-2\xi}\right] - 2\Psi_{(n-2)zz\xi\xi} - \Psi_{(n-4)\xi\xi\xi\xi} + Pr^{-1}\left[\Psi_{(n-4)zz\tau} + \Psi_{(n-6)\xi\xi\tau} + \sum_{m=4}^{4}\frac{\partial(\Psi_{n-m}, \ \Psi_{(m-4)zz} + \Psi_{(m-6)\xi\xi})}{\partial(\xi,z)}\right]$$

$\Psi_n = D\Psi_n = D\theta_n = 0$ on $z = 0$ and $\Box$ (8)

where quantities have the negative or odd subscripts are ignored. By integrating (7) and (8) over the layer depth, the boundary conditions ensure that the left-hand side is zero. Hence the solvability conditions

$$\int_0^{\Box}\left[(3z^2 - 6z + 2)\Psi_{(n-2)\xi} - \theta_{(n-2)\xi\xi} + \theta_{(n-4)\tau} + \sum_{m=4}^{n}\frac{\partial(\Psi_{m-4}, \theta_{n-m})}{\partial(\xi,z)}\right]dz = 0 \quad (9)$$

must be satisfied for all values of n for which $\theta_n$and Ψ_n are known.

At zeroth order the right-hand side of (7) and (8) is zero and hence

$\theta_0 = f_0(\xi,\tau)$

where f_0 is an arbitrary function. Equation (9) can be solved to get

$$\Psi_0 = R_0 f_{0\xi}[T_1 + T_2 + T_3]$$

Where, $T_1 = -\frac{3^7}{7.5!} + \frac{3^6}{5!} - \frac{2z^5}{5!}$

$$T_2 = z^3\left[\frac{□^4}{24.7} + \frac{□^3}{30} - \frac{□^2}{20}\right]$$

$$T_3 = □^2 z^2\left[-4□ + 3□^2 - \frac{4□^3}{7}\right]$$

From the solvability condition (9), we derive the equation for f_0 by taking $□ = 2$

$$□_{0□} + □□_{0□□□□} + □_2 □_{0□□} - □(□_0 □_{0□})_□ = 0$$
(10)

Results and Discussion

The results of the present investigation are presented and discussed through graphs (Figure 3 and 4). The following results are predicted:

i)In figure 3, the graph of $□_{□0}$ vs □ is presented for S-shaped cubic profile. It is observed that, throughout the range $□_{□0} < 0$ only. That is gravitationally unstable fluid layer lies above a thinner stable layer for $□_{□0} > 0$, the inverse situation exists.

ii)In figure 4, the graph of $\Psi_□$ vs □ is presented for different values of □.

In this case, $\Psi_□$ assumes positive values only and for $□ = 0.75$, and $□ = 1.0$ the curves are widely spread. Finally, it is concluded that the results are very encouraging.

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Figures:

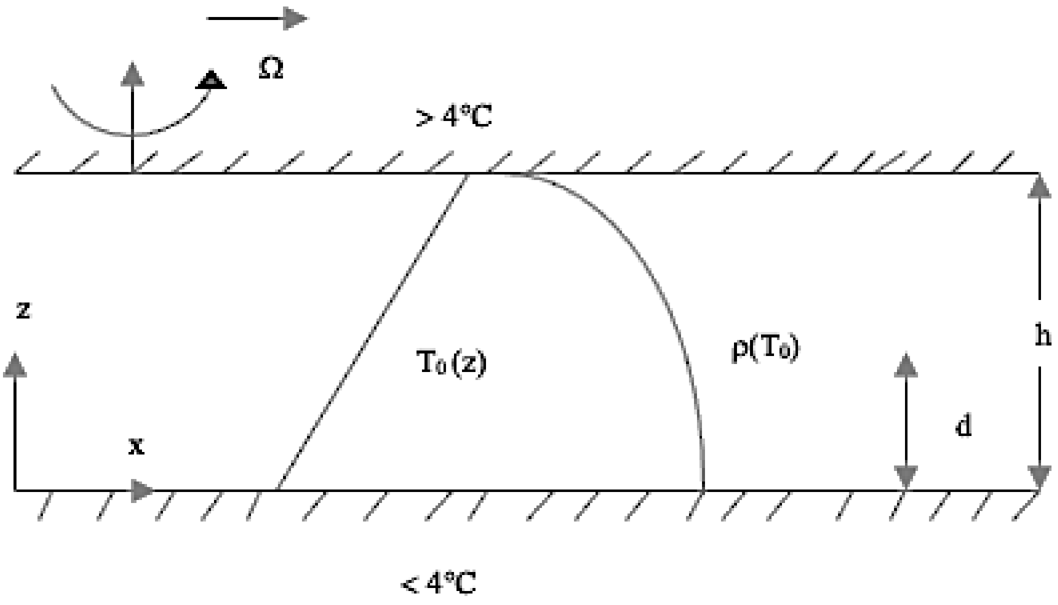

Figure 1: Conductive state in ice-water convection

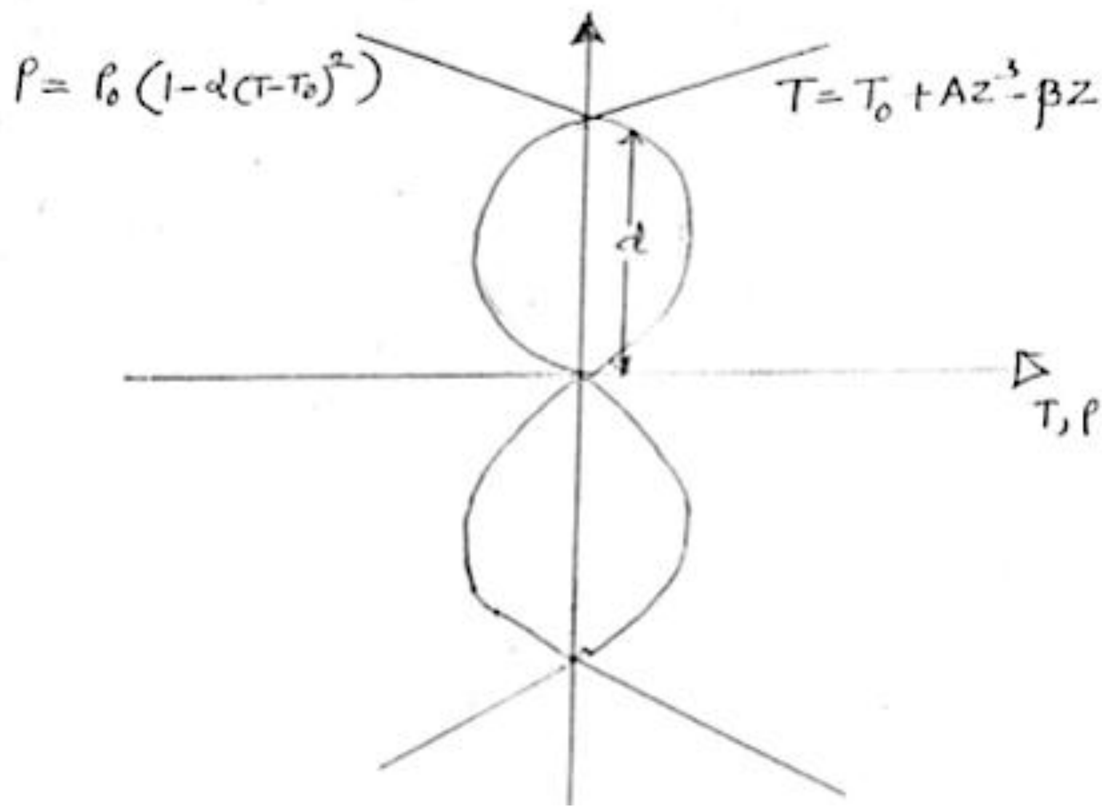

Figure 2: The temperature and density profiles

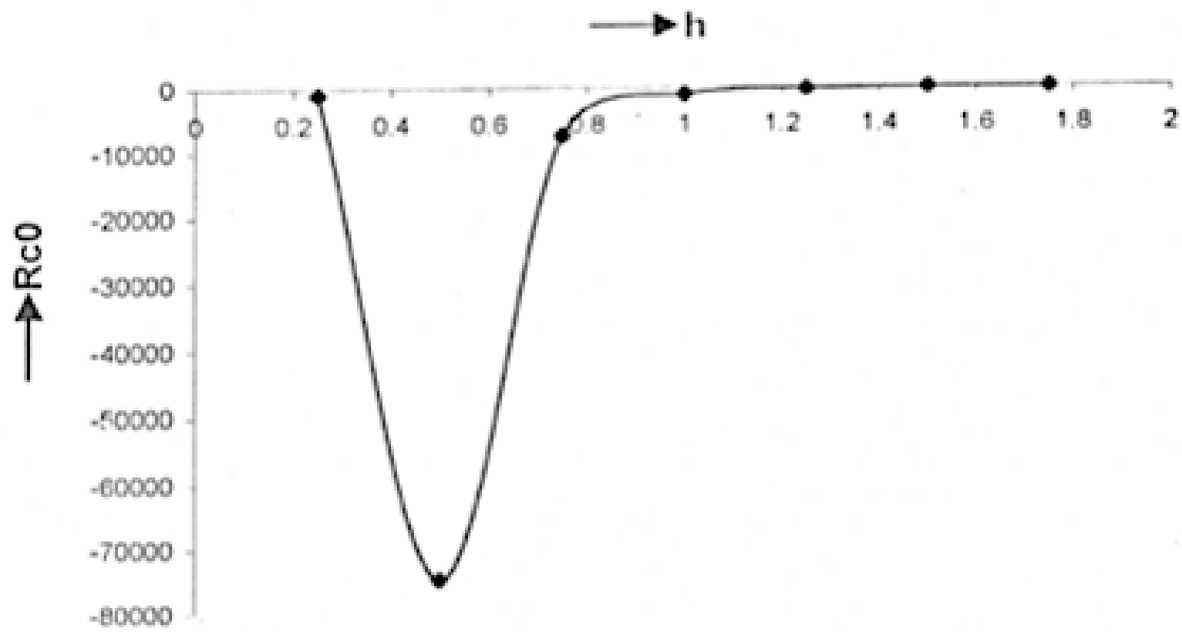

Figure 3: Rc0V/s h for different values of z

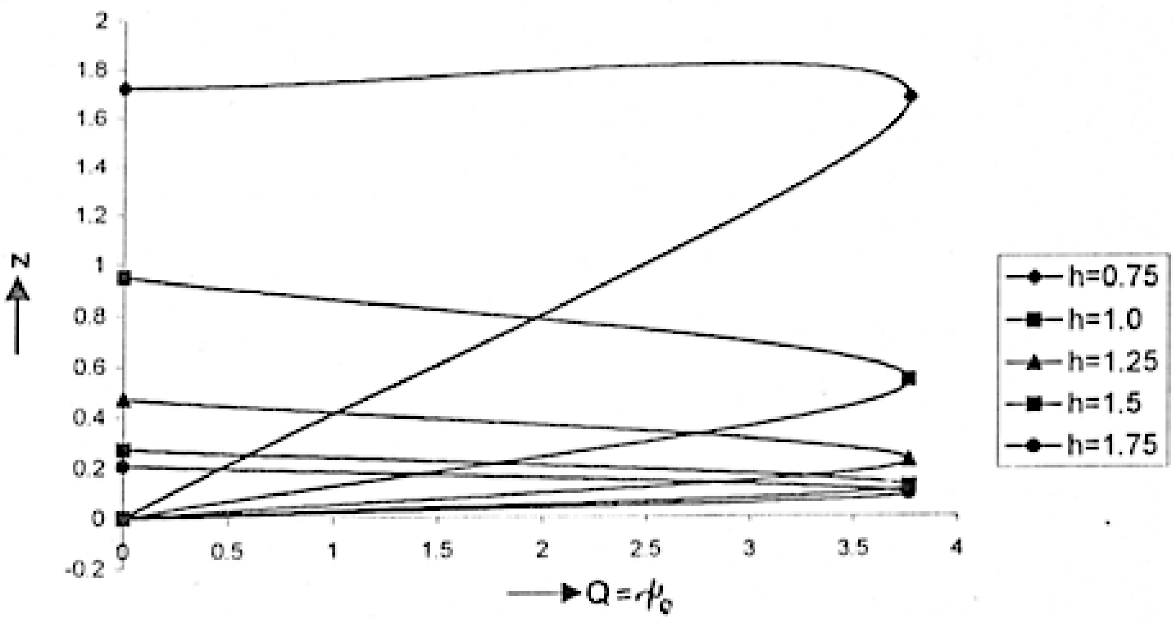

Figure 4: Z V/s Q