International Journal of Advanced Research in Engineering and Applied Sciences ISSN: 2278-6252

THE COMBINATION OF SPIDER GRAPHS WITH STAR GRAPHS FORMS GRACEFUL

Dr. A.N.Mohamad*

Abstract: *A labeled graph G which can be gracefully number is said to be graceful. Labeling the node of G with distinct non negative integers and then labeling the edges of G with the absolute differences between nodes values, if the graph edges numbers runs from $1\ to\ n$, the graph G is gracefully $numbered$. In this paper, we combine some spider graphs with star graphs forming new graphs and show them to be graceful, which have a wide range of applications in different fields such as coding theory, X-ray crystallography, radar, astronomy, circuit design, communication network and data base management systems.*

Keywords: *Graceful, Graph labeling, Spider, Star.*

*Associate Professor, School of Mathematical & Statistical Science, Hawassa University, Hawassa, Ethiopia

1. INTRODUCTION

Graph Theory is a branch of discrete mathematics, distinguished by geometric approach to the study of objects. The principle object of the theory is a graph and its generation. Any problem or object under consideration is represented in the form of nodes (Vertices, points, elements) and edges (lines, link, connections). The vertex set and edge set are denoted by $V(G) and\ E(G)$ respectively in [1]. The Ringel - Kotzig conjectured that all trees are graceful and this has been the focus of many papers [2]

A graceful labeling [3] or numbered f of a graph G with q edges is $f: G \rightarrow \{0,1,2,\dots.q\} is\ an\ injective\ function\ such\ that\ when\ each\ edge\ xy\ is\ assigned$ $the\ label\ |f(x)-f(y)|\ and\ also\ the\ resulting\ labels\ are\ distinct\ and\ non\ zero.$

C.Huang called the effort to prove it a "disease" Among the trees known to be graceful is: Caterpillars (A caterpillar is a tree with a property that the removal of its endpoints leaves the path [5] trees with at most 4 end - vertices [5] [6] [7]

J.C.Bermond conjectured [8] that lobsters are graceful (A lobster is a tree with the property that the removal of the endpoints leaves a caterpillar) Symmetrical trees (A symmetrical tree is a rooted tree in which every level contains vertices of the same degree [8] [9]. Regular bamboo trees (A rooted tree consisting of branches of equal length the end points of which are identified with end points of stars of equal size [10], are graceful.

Olive trees [11] (A rooted tree consisting of $k\ branches\ where\ the\ ith\ branch\ is\ a\ path\ of\ length\ i$) , are graceful.

K.Eshghi & P.Azimi [12] [13] $discussed\ a\ programming\ model\ for\ finding\ graceful\ labelings\ of\ large\ graphs$. The computational results showed that the model can easily solve the graceful labeling problem for large graphs. They used this method to verify that all trees with 30, 35 and 40 vertices are graceful.

D.Morgan investigated that all lobsters with perfect matching are graceful [14] [15] [16]

$Spider\ graphs\ S_{m,n}\ \ (m \geq 1, n \geq$
1have m legs of length n and Stars are Sm, 1 graphs.

$Spider\ graphs\ RE_{m,q}\ \ (m \geq 1, n \geq 1) have\ q\ legs\ of\ length\ 2\ and\ m\ legs\ of\ length\ 1$

In this paper we will form graceful graphs by combining stars with the outer vertices of spider graphs.

1.1. Preliminary assumptions.

- All graphs considered here are finite simple and undirected.
- The vertex set and edge set are denoted by $V(G)$ and $E(G)$. Our notations and terminology are as in[1]
- We $refer$[4] for some basic concepts.

1.2. Basic Definitions

They are needed for further discussion.

Definition - 1:

A graph $G = (V, E)$consists of a finite set denoted by V or $V(G)$, and a collection $E, E(G)$of unordered pairs u, v of distinct elements from V . Each elements of $V$is called a *vertex* or a *point* or a *node* or *element* and each element of E is called an *edge* or a *line* or a *link* or a *connection*. Usually these graphs are also called *simple graphs*.

Definition - 2:

The number of vertices namely the cardinality of V is called the *order* of G and is denoted by $|V|$ or n or p

Definition - 3:

The number of edges of a graph namely the cardinality of E is called the size of G, and is denoted by $|E| or\ m\ or\ q$, we write $e = v_i v_j \in E(G)$to mean that the pair $v_i, v_j \in E(G)$ $when$

$e = v_i v_j \in E(G)$

We say that v_i and v_j are $adjacent, e$ and v_i, e and v_j are $incident$

Definition - 4:

The *open neighborhood* $N(v)$of the vertex v consists of the set of vertices adjacent to v . That is

$N(v) = \{w \in V: vw \in E\}$. For a set $S \subset V$, the open neighborhood $N(S)$is defined by

$N(S) = \cup_{v\in S} N(V)$

The *closed neighborhood* of the vertex v is $N[v] = N(v) \cup \{v\}$and the closed neighborhood $N[S]$ by $N[S] = N(S) \cup S$

Definition - 5:

The *degree of a vertex* v is denoted by $deg(v)$, and defined as the number of edges incident with v that is $deg(v) = \{N(v)\}$. The degree sequence of $G$is $\{degv_1, degv_2, \ldots degv_n\}$, typically written in non - decreasing or non - increasing.

Definition -6:

The maximum and minimum of the degree of vertices of a graph G are denoted by $\Delta(G)$ and $\delta(G)$ respectively.

If $\Delta(G) = \delta(G) = r$ then G $is\ called\ a\ reqular\ graph\ of\ degree\ r\ \ or\ simply\ r-regular$

Note: A $3-regular\ graph\ is\ also\ called\ 3-regular.$

Definition - 7:

A $walk\ of\ length\ \ k$ it is an alternating sequence $W = u_0e_1u_1e_2u_2e_3 \ldots u_{k-1}e_ku_k$ of vertices and edges with $e_i = u_{i-1}u_i$. It is also denoted by $W = u_0u_1u_2 \ldots u_{k-1}u_k$ $(implicitly\ u_iu_{i+1} \in E,\ 0 < i < k-1)$

If all k edges are distinct, then W is called a *trial*.

If all $k+1$ vertices are distinct, then W is called a *path*

If $u_0 = u_k$, then W is called *closed* and closed trial is called a *circuit*.

If $u_0 = u_k$ $and\ u_1u_2 \ldots u_{k-1}\ are\ distinct\ then\ W\ is\ called\ a\ cycle\ or\ k-cycle$

If $u_0 = u$ $and\ u_k = v\ then\ W\ is\ said\ to\ be\ a\ u-v\ walk\ of\ length\ k$

Definition - 8:

A graph is said to be *connected*, if every pair u, v of vertices $G\ there\ exists\ a\ u-v\ path$. Otherwise $G$is said to be *disconnected*

Definition - 9:

If there exists at least one $u - v\ walk$, then the distance $d(u,v)$between $u\ and\ v$, it is the minimum length of $u - v\ walk$; if no $u - v\ walk$exists then it is denoted by $d(u,v) = \infty$

Definition - 10:

The i^{th} open neighborhood of v $for\ i \geq 0$ consists of those vertices at distance i from v and is denoted by $N^i(v)$.

That is, $N^i(v) = \{u \in V: d(u,v) = i\}$

The i^{th} closed neighborhood $N^i[v] = \{u \in V: d(u,v) \leq i\}$

In particular $N^0(v) = N^0[v] = v$, $N^1(v) = N(v)\ and\ also\ N^1[v] = N[v]$

Definition - 11:

The *eccentricity* of a vertex v , denoted by $ecc\ (v)$, is defined as

$ecc\ (v) = max\ \{ecc(v): v \in V\}$

The *radios* of a graph G , denoted by $rad\ (G)$, is defined as

$rad\ (G) = min\{ecc(v): v \in V\}$

The *diameter* of a graph G, denoted by $diam\ (G)$, is defined as

$diam\ (G) = max\{ecc(v): v \in V\}$

The *distance* of a vertex v, denoted by $dist\ (v)$, is defined as

$dist\ (v) = \sum_{u \in V(G)} d(u, v)$

Definition - 12:

The cycle C_n of order $n \geq 3$ has size $m = n(number\ of\ vertices = number\ of\ edges)$ is connected and is 2 - regular.

A tree is a connected graph with no cycle.

The path P_n is the tree of order n with diameter $n - 1$

The star $K_{1,n-1}$, it has one vertex of $(n - 1)$ degree and $(n - 1)$ vertices of degree one.

In any graph, a vertex degree one is called an *end vertex* and an edge incident with and end vertex is called *pendant edge*. Among all connected graphs of order, trees have the minimum number of edges and the *complete graph or clique* (A graph in which any two vertices are adjacent). Any tree is a *bipartite* (A graph whose vertices can be divided into two disjoint sets $U\ and\ V$ such that every edge has one end in U and another end in V). A complete bipartite graph is a special kind of bipartite graph where every vertex of the first set is connected to every vertex of the second set. All complete bipartite graphs are stars when they are trees. A star S_n is the complete bipartite graph $K_{1,n}$

Definition - 13:

Two graphs $G\ and\ H$ are said to be *isomorphic* if there exists a bijection $\chi: V(G) \rightarrow V(H)$ such that $uv \in E(G)$, if and only if $\chi(u)\ \chi(v) \in E(H)$.

1.3. Construction of a graph from two given graphs

Let $G\ and\ H$ are two given graphs then

1. $Union\ \ K = G \cup H\ has\ V(K) = V(G) \cup V(H) and\ E(K) = E(G) \cup E(H)$
2. Intersection, let $V(G) \cup V(H) = \emptyset$. The $join\ K = G + H\ has\ V(K) = V(G)\ \cup$ $V(H)$ and also $E(K) = E(G) \cup E(H) \cup \{uv: u \in V(G)\ and\ v \in (V(H)\}$
3. The Cartesian product $K = G \times H\ \ has\ V(K) = V(G) \times V(H)\ and$ $(u_1, v_1), (u_2, v_2) \in V(K) are\ adjacent\ if\ u_1 = u_2, v_1, v_2 \in E(H)\ and$ $v_1 = v_2, u_1, u_2 \in E(G)$
4. $P_j \times P_k\ is\ called\ the\ grid\ G_{j,k}$
5. $C_j \times C_k\ is\ called\ the\ cylinder\ C_{j,k}$
6. $C_j \times C_k (j, k \geq 3)\ is\ called\ the\ Torus\ T_{j,k}$

2. NECESSARY DEFINITION FOR FURTHER APPROACH

1) A tree T_n on n vertices$v_1, v_2, v_3, \ldots, v_n$ can be represented by,

$T_n = T_n\big(deg(v_1), deg(v_2), \ldots, deg(v_n)\big), where\ deg(v_n)\ is\ the\ nth\ vertex\ in\ T_n$

a. A tree on three vertices is denoted by $T_3 = T_3(1, 1, 2)$

b. Up to isomorphism the trees $T_4\ with\ four\ vertices\ are\ T_4(1, 1, 1, 3)$, $T_4(1, 1, 2, 2)$

c. Up to isomorphism the trees $T_5\ with\ five\ vertices\ are$ $T_5(1, 1, 1, 1,4), T_5(1, 1, 1,2, 3), T_5(1, 1, 1,2, 4)$

d. Up to isomorphism the trees $T_6\ with\ six\ vertices\ are$ $T_6(1, 1, 1, 1,1, 5), T_6(1, 1, 1,1, 2, 4), T_6(1, 1, 1,1, 3, 3), T_6(1, 1,2, 2, 2, 2)$ And two different classes of trees $T_6(1, 1,1, 2, 2, 3)$

2) $S_n\ is\ a\ star\ graph\ with\ (n + 1)\ vertices\ \ and\ n\ legs\ of\ length\ one$

3) Spider graph $S_{m,n}\ is\ defined\ as\ a\ tree\ with\ m\ legs\ of\ length\ n\ \ on\ \ mn + 1\ vertices$

4) $RE_{m,n}\ is\ a\ tree\ with\ m\ legs\ of\ length\ one\ and\ n\ legs\ of\ length\ two$

Let

$o\ is\ an\ outer\ vertex, \alpha\ \ is\ a\ middle\ vertex, c\ is\ the\ center\ vertex\ ad\ p\ is\ the\ pendent\ vertex\ attached\ to\ center$

5) Let $A\ \ and\ B$ are any two graphs. A graph $A * B$ is obtained by gluing a vertex $a(of\ \deg(d))\ of\ the\ graph\ A\ with\ a\ vertex\ b\ (of\ \deg(d))of\ the\ graph\ B$.

a. If a center vertex of A is attached to center vertex of B, then $I$is of category I

b. If a non center vertex of A is attached to center vertex of B, then $A * B$ is of category II

c. If a center vertex of A is attached to non center vertex of B, then $A * B$ is of category III

d. If a non center vertex of A is attached to non center vertex of B, then $A * B$ is of category IV

6) Definition - Category I

If a center vertex of A of any degree is attached to center vertex of B, of any degree then $A * B$ is called $A * B$ of category I, type I. Up to isomorphism, $A * B$ has only one type in category I

7) Definition - Category II

a. If a non center vertex of A of degree one is attached to center vertex of B, of any degree then $A * B$ is called $A * B$ of category II, type I.

b. If a non center vertex of A of degree is two attached to center vertex of B, of any degree then $A * B$ is called $A * B$ of category II, type II.

c. Generally, if a non center vertex of A of degree is m attached to center vertex of B, of any degree then $A * B$ is called $A * B$ of category II, type m.

8) Definition - Category III

a. If a center vertex of A of any degree is attached to non center vertex of B, of degree one then $A * B$ is called $A * B$ of category III, type I.

b. If a center vertex of A of any degree is attached to non center vertex of B, of degree two then $A * B$ is called $A * B$ of category III, type II.

c. Generally, if a center vertex of A of any degree is attached to non center vertex of B, of degree m then $A * B$ is called $A * B$ of category III, type m.

9) Definition - Category IV

a. If a non center vertex of A of degree one is attached to non center vertex of B, of degree one then $A * B$ is called $A * B$ of category IV, type I.

b. If a non center vertex of A of degree two is attached to non center vertex of B, of degree one then $A * B$ is called $A * B$ of category IV, type II.

c. Generally, if a non center vertex of A of degree m is attached to non center vertex of B, of degree one then $A * B$ is called $A * B$ of category IV, type m.

2.1. Combination of spider graphs with one copy of star graph

Basic Theorems:

The combined graph which is $S_{m,2} * S_n$ $of\ category\ II, type\ I\ is\ graceful,\ for$ $m = 3, 4, 5\ and\ positive\ integer\ n \geq 2$

Proof:

Here we will prove this theorem in three cases

<u>**Case-1: For m = 3**</u>

The combined graph which is $S_{3,2} * S_n$ $of\ category\ II, type\ I\ is\ graceful$ $where\ S_{3,2}\ is\ a\ spider\ graph\ with\ seven\ vertices\ and$ $S_n\ is\ a\ star\ with\ (n+1)\ vertices, for\ n \geq 2$

Let us take $v_1, v_2, , \ldots, v_7$ are the vertices of the $S_{3,2}$ $with\ the\ label\ v_1\ denoting$ center *and* v_2, v_4 *and* v_7 *representing the outer vertices and* v_3, v_5, v_6 *representing the inner vertices*

Let us take $v_7, v_8, , \dots, v_{n+7}$ are the vertices of the star S_n

Connect the center of the star S_n is with vertex v_7, and the other n vertices of S_n are named,

By giving the labels $v_8, \dots, v_{n+7}$ continuously starting from one pendant vertex till the last pendant vertex

Now we will get a new graph and is denoted by $S_{3,2} * S_n$, clearly it is a **tree** with

$(n+7)$ *vertices and* $(n+6)$ *edges as shown in the following figure* -1:

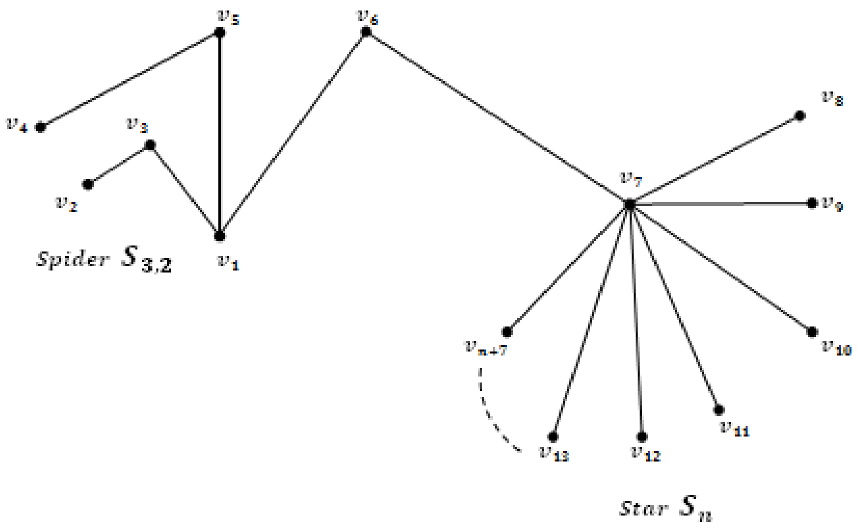

Figure 1. Ordinary labeling of $S_{3,2} * S_n$ *of category II, type I*

The labeling f on the vertices of $S_{3,2} * S_n$ is defined by

$f(v_i) = i \qquad for\ i = 1, 3, 6, 7$

$f(v_i) = i + 2 \qquad for\ i = 2$

$f(v_i) = i - 2 \qquad for\ i = 4$

$f(v_i) = i - 1 \qquad for\ 8 \le i \le n + 7$

Then, the labeling f on the vertices of $S_{3,2} * S_n$ *of category II, type I*, satisfies the conditions for a graceful labeling

Hence complete the proof of case - 1.

<u>Case - 2: For $m = 4$</u>

The combined graph which is $S_{4,2} * S_n$ *of category II, type I is graceful,*

where $S_{4,2}$ *is a spider graph with nine vertices and* S_n *is a star with*

$(n+1)$ *vertices, for* $n \ge 2$

Let us take $v_1, v_2, , \dots, v_9$ are the vertices of the spider $S_{4,2}$ $with\ the\ label\ v_1\ denoting\ the\ centerand\ v_2, v_4, v_6\ and\ v_9\ representing\ the\ outer\ vertices\ and\ v_3, v_5, v_{7,}\ v_8\ representing\ the\ inner\ vertices$

Let us take $v_{10}, v_{11}, \dots, v_{n+9}$ are the vertices of the star S_n

Connect the center of the star S_n is with vertex v_9, and the other n vertices of S_n are named, By giving the labels $v_{10}, v_{11}, \dots v_{n+9}$ continuously starting from one pendant vertex till the last pendant vertex

Now we will get a new graph and is denoted by $S_{4,2} * S_n$, clearly it is a **tree** with $(n + 9\ vertices\ and$ $(n + 8)\ edges\ as\ shown\ in\ the\ following\ figure - 2$:

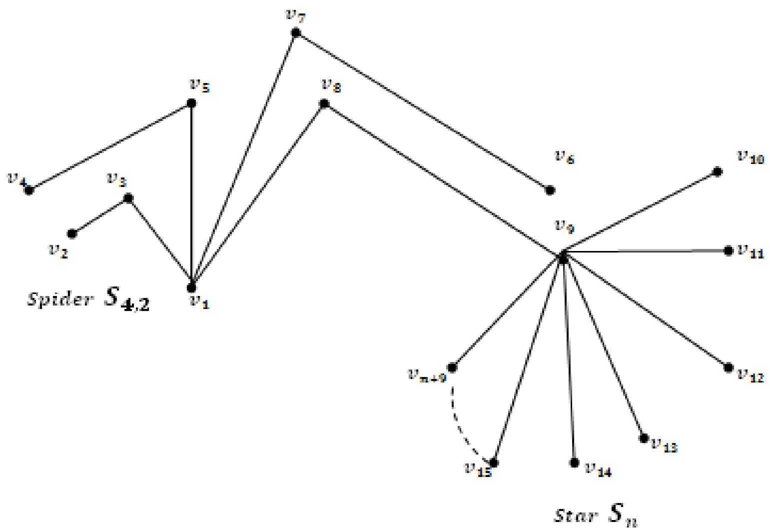

Figure 2. Ordinary labeling of $S_{4,2} * S_n\ of\ category\ II, type\ I$

The labeling f on the vertices of $S_{4,2} * S_n$ is defined by

$f(v_i) = 0 \qquad for\ i = 1$

$f(v_i) = i + 1 \qquad for\ i = 2$

$f(v_i) = i + n + 2 \qquad for\ i = 3, 4, 5$

$f(v_i) = 1 \qquad for\ i = 6$

$f(v_i) = i + n - 3 \qquad for\ i = 7$

$f(v_i) = i + n \qquad for\ i = 8$

$f(v_i) = i - 7 \qquad for\ i = 9$

$f(v_i) = i - 6 \qquad for\ 10 \le i \le n + 9$

Then, the labeling f on the vertices of $S_{4,2} * S_n$ $of\ category\ II, type\ I$, satisfies the conditions for a graceful labeling

Hence complete the proof of case - 2.

<u>Case - 3: For $m = 5$</u>

The combined which is $S_{5,2} * S_n\ of\ category\ II, type\ I\ is\ graceful$ graph

$S_{5,2} *, where\ S_{5,2}\ is\ a\ spider\ graph\ \ with\ eleven\ vertices\ and\ S_n\ is\ a\ star\ with$

$(n+1)\ vertices, for\ n \geq 2$

Let us take $v_1, v_2, v_3, \ldots, v_{11}$ are the vertices of the

spider $S_{5,2}$ $with\ the\ label\ v_1\ denoting\ the$

$center\ and\ v_2, v_4, v_6\ , v_8\ and\ v_{11}\ representing\ the\ outer\ vertices\ and\ v_3, v_5, v_{7,}\ v_{9,}\ v_{11}$

$representing\ \ the\ inner\ vertices$

Let us take $v_{12}, v_{13}, \ldots, v_{n+11}$ are the vertices of the star S_n

Connect the center of the star S_n is with vertex v_9, and the other n vertices of S_n are named,

By giving the labels $v_{10}, v_{11}, \ldots v_{n+9}$ continuously starting from one pendant vertex till the last pendant vertex

Now we will get a new graph and is denoted by $S_{5,2} * S_n$, clearly it is a **tree** with $(n +$

$11\ vertices\ and$

$(n+10)\ \ edges\ as\ shown\ in\ the\ following\ figure - 3$:

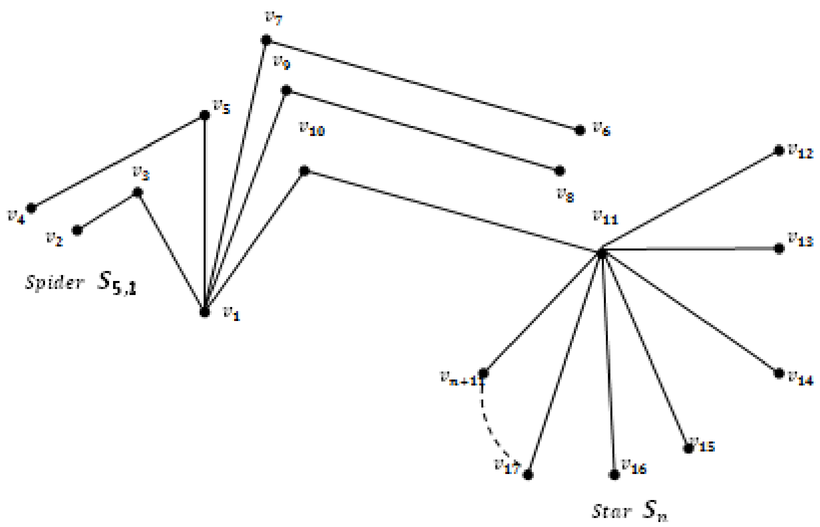

Figure 3. Ordinary labeling of $S_{5,2} * S_n\ of\ category\ II, type\ I$

The labeling f on the vertices of $S_{4,2} * S_n$ is defined by

$f(v_i) = 0 \qquad\qquad for\ i = 1$

$f(v_i) = i \qquad for\ i = 2, 5, 7$

$f(v_i) = i + 3 \qquad for\ i = 3$

$f(v_i) = i - 1 \qquad for\ i = 4, 9$

$f(v_i) = i - 2 \qquad for\ i = 6$

$f(v_i) = i + 1 \qquad for\ i = 8$

$f(v_i) = i + n \qquad for\ i = 10$

$f(v_i) = 1 \qquad for\ i = 11$

$f(v_i) = i - 2 \qquad for\ 12 \leq i \leq n + 11$

Then, the labeling f on the vertices of $S_{5,2} * S_n$ $of\ category\ II, type\ I$, satisfies the conditions for a graceful labeling

Hence complete the proof of case - 3.

2.2. Combination of general graphs $RE_{m,q}$ of category II, type I with one copy of star graphs

Theorem -2:

For $n \geq 1, m \geq 1\ and\ q \geq 1, the\ graph\ RE_{m,q} * S_n$, of category II, type I, then it is graceful if $RE_{m,q}$ is spider graph on $(m + 2q + 1)$, vertices and also S_n is a star graph with $(n + 1)$

Vertices.

Proof:

Before we prove this theorem, we can prove the following theorems

Theorem - 2 a:

For $n \geq 2, m \geq 2$, $the\ graph\ RE_{m,1} * S_n$, of category II, type I, then it is graceful if $RE_{m,1}$ is spider graph on $(m + 3)$, vertices and also S_n is a star graph with $(n + 1$ vertices.

Proof:

Let $RE_{m,1}$ is a tree with m legs of length one and one leg with length two.

Let $v_1, v_2, v_3, \dots, v_m , v_{m+2}$ are the pendent vertices of the tree $RE_{m,1}$ with the label v_{m+3} has degree

of $(m + 1)$ $and\ the\ label\ v_{m+1}$ $representing\ the\ vertex\ adjacent\ to\ v_{m+2}\ and\ v_{m+3}$ $of\ degree\ 2$

The star has$(n + 1)$vertices they have been labeled by giving the labels $w_1, w_2, w_3, \dots, w_{n+1}$

Continuously starting from one pendent vertex till the last pendent vertex and take the label $w_1$for the center of S_n

To form the graph $RE_{m,1} * S_n$, of category II, type I

First the vertex w_1 $of\ S_n\ is\ merged\ with\ the\ vertex\ v_{m+2}\ of\ \ RE_{m,1}\ and\ labled\ as\ v_{m+2}$

There are $(m + n + 3)$ vertices and $(m + n + 2)$ edges

The name of the

labels $w_2, w_3, \ldots, w_{n+1}$ $are\ changed\ as\ v_{m+4}, v_{m+5}, \ldots, v_{m+n+3}\ repectively.$

$As\ shown\ in\ the\ following\ figure - 4$:

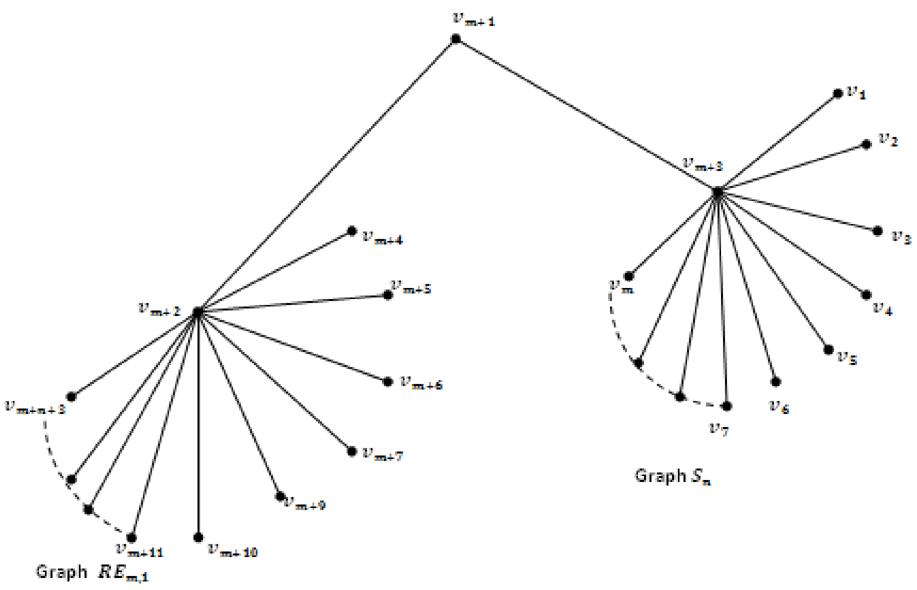

Figure 4. Ordinary labeling of $\boldsymbol{RE_{m,1} * S_n\ of\ category\ II, type\ I}$

The labeling f on the vertices of $RE_{m,1} * S_n$ is defined by

$f(v_i) = i \qquad for\ \ 1 \le i \le m$

$f(v_i) = i - m - 1 \qquad for\ i = m + 1$

$f(v_i) = i - 1 \qquad for\ i = m + 2$

$f(v_i) = i + n - 1 \qquad for\ i = m + 3$

$f(v_i) = i - 2 \qquad for\ m + 4 \le i \le m + n + 3$

Then, the labeling f on the vertices of $RE_{m,1} * S_n$ $of\ category\ II, type\ I$, satisfies the conditions for a graceful labeling

Hence complete the proof of theorem -2 a.

Theorem - 2 b:

For $n \geq 1, m \geq 1$, *the graph* $RE_{m,2} * S_n$, of category II, type I, then it is graceful if $RE_{m,2}$ is spider graph on $(m+5)$, vertices and also S_n is a star graph with $(n+1)$ vertices.

Proof:

Let $RE_{m,2}$ is a tree with m legs of length one and two legs with length two.

Let v_1 is the center and b, c *are middle vertices and* d, e *are the outer vertices and* $v_2, v_3, \ldots, v_m, v_{m+1}$, are single vertices of $RE_{m,2}$

The star has $(n+1)$ vertices they have been labeled by giving the labels $v_{m+6}, \ldots, v_{m+n+5}$ Continuously starting from one pendent vertex till the last pendent vertex and take the label v for the center of S_n

To form the graph $RE_{m,2} * S_n$, of category II, type I

First the

vertex v *of* S_n *is merged with the vertex* e *of* $RE_{m,2}$ vertex

adjacent to c *and labled as*

v_{m+5}. There are $(m+n+5)$ vertices and $(m+n+4)$ edges

The name of the labels b, c *and* d *are changed as* $v_{m+2}, v_{m+3},$ *and* v_{m+4} *repectively.*

As shown in the following figure $-$ 5

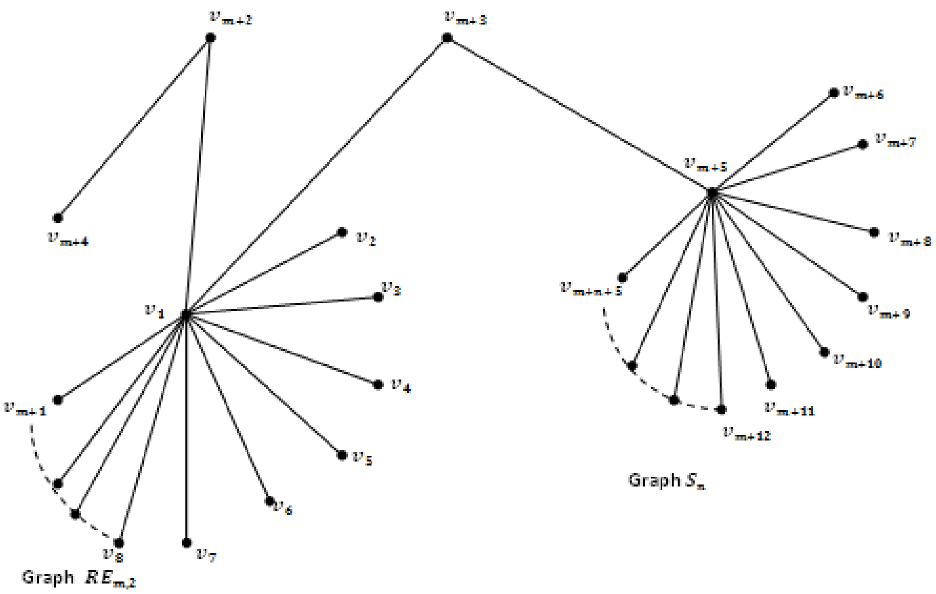

Figure 5. Ordinary labeling of $RE_{m,2} * S_n$ ***of category II, type I***

The labeling f on the vertices of $RE_{m,2} * S_n$ is defined by

$f(v_i) = 1 \qquad for\ i = 1$

$f(v_i) = i + n + 2 \qquad for\ 2 \leq i \leq m + 2$

$f(v_i) = n + 3 \qquad for\ i = m + 3$

$f(v_i) = 0 \qquad for\ i = m + 4$

$f(v_i) = i - m - 3 \qquad for\ m + 5 \leq i \leq m + n + 5$

Then, the labeling f on the vertices of $RE_{m,2} * S_n\ of\ category\ II, type\ I$, satisfies the conditions for a graceful labeling

Hence complete the proof of theorem -2 b.

Theorem - 2 c:

For $n \geq 1, m \geq 1$, $the\ graph\ RE_{m,3} * S_n$, of category II, type I, then it is graceful if $RE_{m,3}$ is spider graph on $(m + 7)$, vertices and also S_n is a star graph with $(n + 1$ vertices.

Proof:

Let $RE_{m,4}$ is a tree with m legs of length one and three legs with length two.

Let v_1 is the center

and $b, c, d\ are\ middle\ vertices\ and\ e, y, g\ are\ the\ outer\ vertices\ and$

$v_2, v_3, \ldots, v_m, v_{m+1}$, are single vertices of $RE_{m,3}$

The star has $(n + 1)$ vertices they have been labeled by giving the labels $v_{m+10}, \ldots, v_{m+n+9}$ Continuously starting from one pendent vertex till the last pendent vertex and take the label v for the center of S_n

To form the graph $RE_{m,3} * S_n$, of category II, type I

First the

vertex $v\ of\ S_n\ is\ merged\ with\ the\ vertex\ g\ of\ RE_{m,3}\ adjacent\ to\ d\ and\ labled\ as$ v_{m+7} . There are $(m + n + 7)$ vertices and $(m + n + 6)$ edges

The name of the labels $b, c\ , d, e\ and\ y\ are\ changed\ as\ v_{m+2}, v_{m+3}, v_{m+4}, v_{m+5}\ and\ v_{m+6}$ $repectively. As\ shown\ in\ the\ following\ figure - 6$:

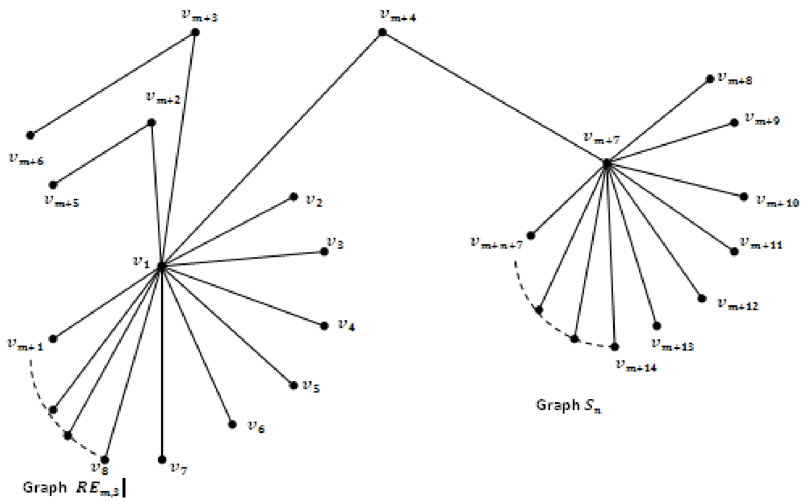

Figure 6. Ordinary labeling of $RE_{m,3} * S_n$ *of category II, type I*

The labeling f on the vertices of $RE_{m,3} * S_n$ is defined by

$f(v_i) = 1 \qquad for\ i = 1$

$f(v_i) = i + 1 \qquad for\ 2 \leq i \leq m + 2$

$f(v_i) = m + n + 6 \qquad for\ i = m + 3$

$f(v_i) = m + n + 5 \qquad for\ i = m + 4$

$f(v_i) = m + 4 \qquad for\ i = m + 5$

$f(v_i) = 0 \qquad for\ i = m + 6$

$f(v_i) = 2 \qquad for\ i = m + 7$

$f(v_i) = i - 3 \qquad for\ m + 8 \leq i \leq m + n + 7$

Then, the labeling f on the vertices of $RE_{m,3} * S_n$ $of\ category\ II, type\ I$ satisfies the conditions for a graceful labeling. Hence complete the proof of theorem -2 c.

Theorem - 2 d:

For $n \geq 1, m \geq 1$, $the\ graph\ RE_{m,4} * S_n$, of category II, type I, then it is graceful if $RE_{m,4}$ is spider graph on $(m + 9)$, vertices and also S_n is a star graph with $(n + 1)$ vertices.

Proof:

Let $RE_{m,4}$ is a tree with m legs of length one and four legs with length two.

Let v_1 is the center

and $b, c, d\ , e\ are\ middle\ vertices\ and\ y, g, h, i\ are\ the\ outer\ vertices\ and$

$v_2, v_3, \ldots, v_m\ , v_{m+1}$, are single vertices of $RE_{m,4}$

The star has $(n+1)$vertices they have been labeled by giving the labels $v_{m+10}, \dots, v_{m+n+9}$ Continuously starting from one pendent vertex till the last pendent vertex and take the label v for the center of S_n

To form the graph $RE_{m,4} * S_n$, of category II, type I

First the

vertex $v\ of\ S_n\ is\ with\ the\ outer\ vertex\ i\ of\ \ RE_{m,4}\ adjacent\ to\ e\ and\ labled\ v_{m+9}$

There are $(m+n+9)$ vertices and $(m+n+8)$ edges

The name of the labels $b, c\ , d\ \ e, y, g\ and\ h\ \ are\ changed\ as\ v_{m+2}, v_{m+3}, v_{m+4},$ $v_{m+5}\ , v_{m+6}, \quad v_{m+7}\ and\ \ v_{m+8}\ repectively.$

The labeling f on the vertices of $RE_{m,4} * S_n$ is defined by

$f(v_i) = 1 \qquad for\ i = 1$

$f(v_i) = 4 \qquad for\ i = 2$

$f(v_i) = i + n + 6 \qquad for\ \ 3 \le i \le m + 1$

$f(v_i) = 3 \qquad for\ i = m + 2$

$f(v_i) = 6 \qquad for\ i = m + 3$

$f(v_i) = m + n + 8 \qquad for\ i = m + 4$

$f(v_i) = n + 8 \qquad for\ i = m + 5$

$f(v_i) = 7 \qquad for\ i = m + 6$

$f(v_i) = 5 \qquad for\ i = m + 7$

$f(v_i) = 0 \qquad for\ i = m + 8$

$f(v_i) = 2 \qquad for\ i = m + 9$

$f(v_i) = i - m - 2 \qquad for\ m + 10 \le i \le m + n + 9$

Then, the labeling f on the vertices of $RE_{m,4}\ * S_n\ \ of\ category\ II, type\ I$, satisfies the conditions for a graceful labeling

Hence complete the proof of theorem -2 d. The graph is shown in the following figure - 7.

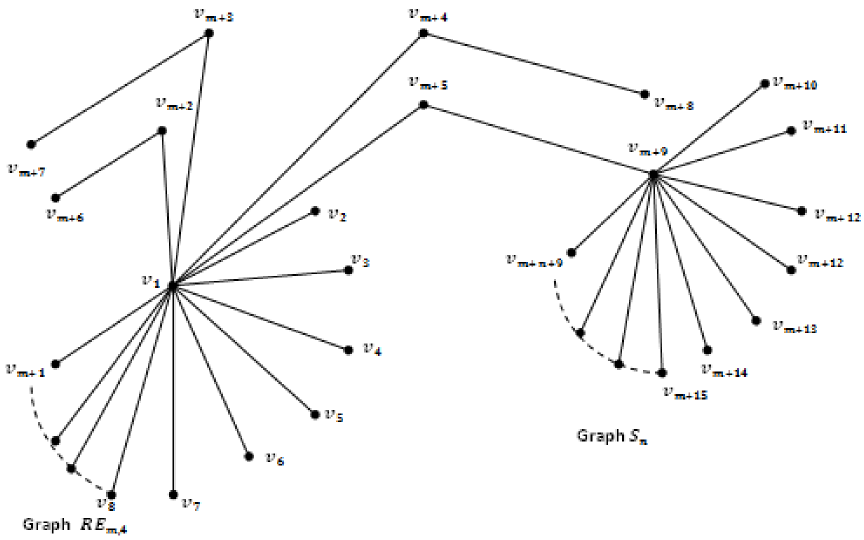

Figure 7. Ordinary labeling of $RE_{m,4} * S_n$ $of\ category\ II, type\ I$

Theorem - 2 e:

For $n \geq 1, m \geq 1$, $the\ graph\ RE_{m,5} * S_n$, of category II, type I, then it is graceful if $RE_{m,5}$ is spider graph on $(m + 11)$, vertices and also S_n is a star graph with $(n + 1)$ vertices.

Proof:

Let $RE_{m,5}$ is a tree with m legs of length one and five legs with length two.

Let v_1 is the center

and b, c, d, e, y $are\ middle\ vertices\ and\ g, h, i, j, k\ are\ the\ outer\ vertices$

$and\ v_2, v_3, \ldots, v_m, v_{m+1}$, are single vertices of $RE_{m,5}$

The star S_n has $(n + 1)$ vertices which is tagged by giving the labels $v_{m+12}, \ldots, v_{m+n+11}$ Continuously starting from one pendent vertex till the last pendent vertex and take the label v for the center of S_n

To form the graph $RE_{m,4} * S_n$, of category II, type I

First the

vertex

$v\ of\ S_n\ is\ combined\ with\ the\ pendent\ vertex\ k\ of\ RE_{m,5}\ which\ is\ adjacent\ to\ y$ $and\ labled\ as\ v_{m+11}$. There are $(m + n + 11)$ vertices and $(m + n + 10)$ edges

The name of the labels $b, c\ , d\ e, y, g, h, i\ and\ j\ are\ changed\ as\ v_{m+2}, v_{m+3}, v_{m+4},$

$v_{m+5}\ , v_{m+6},$

$v_{m+7}\ , v_{m+8}, v_{m+9}\ and\ \ v_{m+10}\ repectively. As\ shown\ in\ the\ following\ figure-8$:

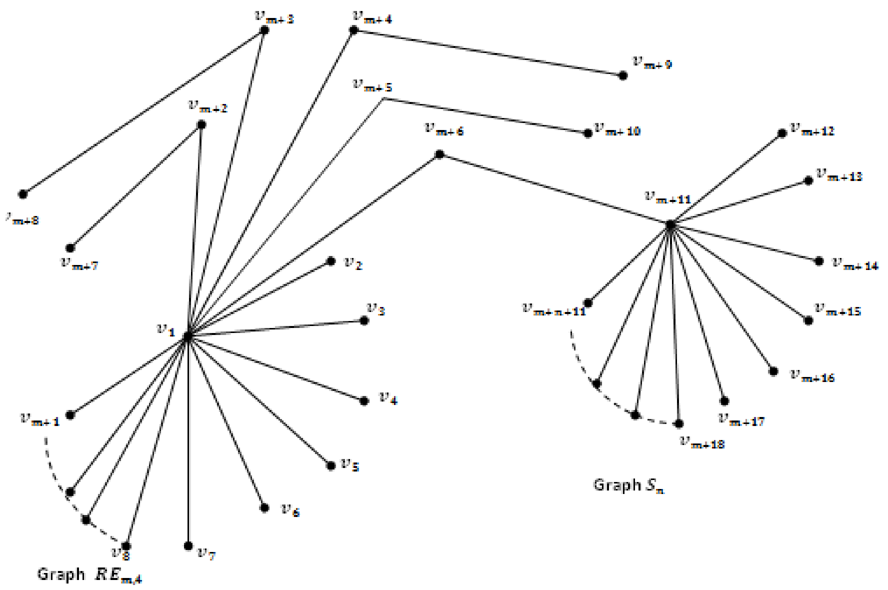

Figure 8. Ordinary labeling of $RE_{m,5} * S_n$ $of\ category\ II, type\ I$

The labeling f on the vertices of $RE_{m,4} * S_n$ is defined by

$f(v_i) = 1 \qquad for\ i = 1$

$f(v_i) = 4 \qquad for\ i = 2$

$f(v_i) = 1 \qquad for\ \ i = 1$

$f(v_i) = i + n + 8 \qquad for\ 2 \le i \le m + 1$

$f(v_i) = 6 \qquad for\ i = m + 2$

$f(v_i) = 7 \qquad for\ i = m + 3$

$f(v_i) = 4 \qquad for\ i = m + 4$

$f(v_i) = m + n + 10 \quad for\ i = m + 5$

$f(v_i) = n + 9 \qquad for\ i = m + 6$

$f(v_i) = 8 \qquad for\ i = m + 7$

$f(v_i) = 3 \qquad for\ i = m + 8$

$f(v_i) = 5 \qquad for\ i = m + 9$

$f(v_i) = 0 \qquad for\ i = m + 10$

$f(v_i) = 2 \qquad for\ i = m + 11$

$f(v_i) = i - m - 3 \qquad for\ m + 12 \le i \le m + n + 11$

Then, the labeling f on the vertices of $RE_{m,5} * S_n$ $of\ category\ II, type\ I$, satisfies the conditions for a graceful labeling

Hence complete the proof of theorem -2 e.

The proof of Theorem -2:

First, the graph $RE_{m,q}$, it is defined as having center node, q legs of length 2 and m legs of length 1.

In this graph there are three classes of edges and four classes of vertices.

These will be referred to as the classes of *outer*, *inner* and *single* edges and *center*, *middle*, *single* and *outer* vertices respectively.

A leg of length one has a single edge which is incident with the center vertex and a single vertex.

The inner edge is incident with the center vertex and a middle vertex and the outer edge is incident with the middle vertex and an outer vertex.

A star graph S_n, it has a center node and n legs of length one.

Then, the center node of the star S_n is attached with an outer vertex of the graph $RE_{m,q}$ to form $RE_{m,q} * S_n$. This is a tree with vertices $(m + 2q + n + 1)$ and edges $(m + 2q + n)$

Secondly, let $v_1, v_2, v_3, \dots, v_{m+2q+1}$ are the vertices of the spider graph $RE_{m,q}$ with the label $v_1$represents the center and $v_2, v_3, \dots, v_{m+2q+1}$are representing the single vertices and $v_{m+2}, v_{m+3}, v_{m+4}, \dots, v_{m+q+1}$, denoting the middle vertices

and $v_{m+q+2}, v_{m+q+3}, \dots, v_{m+2q+1}$ indicating the outer vertices:

Thirdly, the center of the star S_n is affixed with the outer vertex v_{m+2q+1} and the other vertices of are labeled, by giving the labels $v_{m+2q+2}, v_{m+2q+3}, \dots, v_{m+2q+n+1}$continuously starting from one pendent vertex till the last pendant vertex. The graph of $RE_{m,q} * S_n\ as\ shown\ in\ the\ following\ figure - 9$:

The labeling on the vertices of $RE_{m,q} * S_n$ $of\ category\ II, type\ I$ is defined by

$f(v_i) = 1$ $\quad for\ i = 1$

$f(v_i) = 2q + n - 2 + i$ $\quad for\ 2 \le i \le m + 1$

$f(v_i) = 2i - 2m - 1$ $\quad for\ m + 2 \le i \le m + q - 1$

$f(v_i) = i + q + n$ $\quad for\ i = m + q$

$f(v_i) = i + q + n - m - 2$ $\quad for\ i = m + q + 1$

$f(v_i) = 2m + 4q - 2i + 2$ $\quad for\ m + q + 2 \le i \le m + 2q - 1$

$f(v_i) = i - m - 2q \qquad for\ i = m + 2q$

$f(v_i) = i - m - 2q + 1 \qquad for\ i = m + 2q + 1$

$f(v_i) = i - m - 3 \qquad for\ m + 2q + 2 \leq i \leq m + 2q + n + 1$

Then, the labeling f on the vertices of $RE_{m,q} * S_n$ $of\ category\ II, type\ I$, satisfies the conditions for a graceful labeling.

Corollary - 2 - 1:

The graph $S_{q,2} * S_n$ of category II, type I is graceful for $q \geq 1\ and\ n \geq 1$

Proof:

In theorem -2, by putting $m = 0$, the graph $S_{q,2} * S_n$ of category II, type I is constructed.

Case - 1: $q = 1$

The labeling f on the vertices $S_{1,2} * S_n$ of category II, type I it is defined by

$f(v_i) = 1 \qquad for\ i = 1$

$f(v_i) = 1 + n \qquad for\ i = 2$

$f(v_i) = i - 2 \qquad for\ 4 \leq i \leq n + 3$

Then, the labeling f on the vertices of $S_{1,2} * S_n$ $of\ category\ II, type\ I$ satisfies the conditions for a graceful labeling.

Case -2: $q > 1$

Put $m = 0$, in the labeling f defined in theorem -2 to get a graceful labeling of the graph $S_{q,2} * S_n$ of category II, type I.

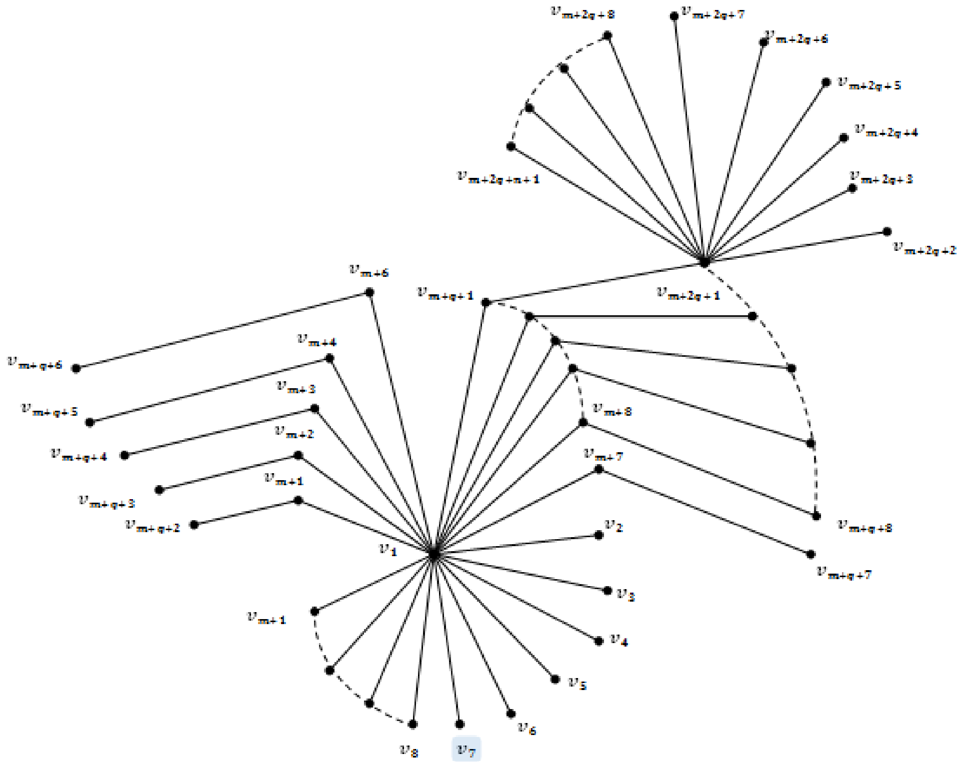

Figure 9. Ordinary labeling of $RE_{m,q} * S_n$ *of category II, type I*

<u>Corollary - 2 - 2:</u>

The graph $RE_{m,q}$, it is graceful for $q \geq 1$ *and* $m \geq 1$

Proof:

Put $n = 0$, in the theorem - 2 to construct the graph $RE_{m,q}$

Case -1: q = 1

The labeling f on the vertices of $RE_{m,1}$, it is defined by

$f(v_i) = i$ $\quad$ *for* $1 \leq i \leq m+2$

$f(v_i) = i - m - 3$ $\quad$ *for* $i = m+3$

Then, the labeling f on the vertices of $RE_{m,1}$ satisfies the conditions for a graceful labeling.

Case - 2: q = 2

The labeling f on the vertices of $RE_{m,2}$, it is defined by

$f(v_i) = i$ $\quad$ *for* $i = 1$

$f(v_i) = i + 2$ $\quad$ *for* $2 \leq i \leq m+2$

$f(v_i) = i - m$ $\quad$ *for* $i = m+3$

$f(v_i) = i - m - 4$ $\quad$ *for* $i = m+4$

$f(v_i) = i - m - 3$ $\quad$ *for* $i = m+5$

Then, the labeling f on the vertices of $RE_{m,1}$ satisfies the conditions for a graceful labeling.

Case - 3: q > 2

Put $n = 0$, in the labeling f defined in theorem -2 to get a graceful labeling of the graph $RE_{m,q}$

REFERENCES

1. A. Rosa, *"On certain valuations of the vertices of a graph"*. Theory of Graphs (international Symposium, Rome, July 1966) Gordon and N.Y. Breach and Dunod Paris, (1978) pp: 349 - 355.
2. G.Ringel and A. Llado, *"Another tree Conjecture"*, Bull.Inst.Combin.Appl.,Vol 18,(1982), pp 83-85
3. J.A. Gallian, "A dynamic survey of graph labeling" The electronic Journal of combinatorics, Vol. 14 (2007), 7 - 28.
4. G. Ore, "Theory of Graphs" (1962), Pearson publications.
5. C. Huang, A. Kotzig, and A. Rosa *"Further results on tree labelings"* Utill.math., Vol.21c,(2001),pp 31-48.
6. K.M. Koh, T. Tan and D.G. Rogers, *"Two theorems on graceful trees"* Discrete math., Vol.25,(1979),pp 141-148.
7. K.M. Koh, T. Tan and D.G. Rogers, *"Another class of graceful trees"* J.Austral Math. Soc.Ser.A,Vol.31 (1981),pp 226-235.
8. J.C. Bremond, *"Graceful graphs, radio antennae and French windmills"* Graph Theory and combinatorics, Pitman, London (1979), pp 18 - 37.
9. J.C. Bremond and D. Sotteau, *"Graphs Decomposition and G - Design"*, Proc.5th Brittish combin.Conf.,(1979),pp 18-37.
10. C. Sekar *"Studies in Graph Theory"*, PhD Thesis, (2002), Madurai Kamaraj University, Tamil Nadu, India.
11. V.J. Abhyanker and V.N. Bhatt - Nayak, *"Easiest Graceful labeling of Olive Trees"* Bull.,Bombay Math.Coll.,Vol.47 (2004),pp 297-309.
12. K.Eshghi, and P.Azimi, *"Applications of mathematical programming in graceful labeling of graph" Journal of Applied Math.,Vol.1 (2004),pp 1-8.*
13. K.Eshghi, and P.Azimi, *"An Algorithm for finding a feasible solution of graph labeling problems" Util.Math.,Vol.72 (2007),pp 163-174.*
14. D.Morgan, *"Gracefully labeled from Skolem sequences"* Congr.Numer., Vol.142 (2000),pp 41-48.

15. D. Morgan, *"All lobsters with perfect matching are graceful"*, Electron,Notes Discrete Math.,Vol 11 (2002) pp 6.
16. D. Morgan and R. Rees, *" Using Skolem and Hooked - Skolem sequences to generate graceful trees"* J.Combin.math.Combin.Comput.,Vol.44 (2003),pp 47-63.