International Journal of Advanced Research in Engineering and Applied Sciences

ISSN: 2278-6252
Impact Factor: 7.358

EFFECT AND EFFICIENT APPROACH OF SOLVING HEAT EQUATION BY DIFFERENT NUMERICAL APPROACHES INCLUDING CRANK – NICOLSON SCHMIDT (METHOD) WHICH IS STILL ACTING AS A MAJOR ROLE

A. N. Mohamad, PhD, Professor, Department of Mathematics, Debre Berhan University, Debre Berhan, Ethiopia

Sisay Fikadu, M. Sc (Student), Department of Mathematics, Debre Berhan University, Debre Berhan, Ethiopia

Abstract: *In this paper we discuss the equation that governs the conduction of heat through a body. We begin with the "Heat equation" (HE) in one – dimensional case in a bounded domain, by the Fourier's Point of view,by numerical approach using Finite Differences Methods (FDM), Bender – Schmidt recurrence equation (BSRE), by the view ofRadial Function, andby Implicit Collocation Technique (ICT), finally theCrank Nicolson Schmidt (CNS) solving partial Differential equation (HE Only), by using an example and get the reasonably coincide approximate solution compare with exact solution and also we explain CNS is still useful like as a major role to solve HE. Discuss with numerical example for each methods.*

Keywords: *One dimensional heat equation, Explicit Scheme, Implicit Scheme, Classification*

1. INTRODUCTION

Partial differential equation (PDE) occurs in many branches of applied mathematics and Engineering mathematics. It appears in description of physical process, for example Hydrodynamics, elasticity,quantum mechanics, and electromagnetic theory etc. The solution of the equation describes possible physical reactions that have to be fixed through boundary conditions, which may be quite a different character. We shall confine ourselves to second order partial differential equations as the equations are found in the applications. First we classify the partial differential equations, which is in category of parabolic equation.

1.1. Classification of the partial differential equations – Basic Concepts:

The general linear partial differential equation in the second derivatives is of the form

$$\left.\begin{aligned} &A\frac{\partial^2 u}{\partial x^2}+B\frac{\partial^2 u}{\partial x\partial y}+C\frac{\partial^2 u}{\partial y^2}+D\frac{\partial u}{\partial x}+E\frac{\partial u}{\partial y}+Fu=G \text{ Or}\\ &Au_{xx}+Bu_{xy}+Cu_{yy}+Du_x+Eu_y+Fu=G \end{aligned}\right\} \longrightarrow (1.1)$$

$$Where\ A, B, C, D, E\ and\ F\ \ are\ all\ functions\ of\ x\ and\ y$$

Note:

For our convention in PDE we can denote,

$$p = \frac{\partial u}{\partial x}; q = \frac{\partial u}{\partial y}; r = \frac{\partial^2 u}{\partial x^2}; s = \frac{\partial^2 u}{\partial x \partial y}; and t = \frac{\partial^2 u}{\partial y^2}$$

And also we have

$$du = \frac{\partial u}{\partial x} dx + \frac{\partial u}{\partial y} y$$

$$\Longrightarrow du = pdx + qdy$$

$$dp = d\left[\frac{\partial u}{\partial x}\right] = \left(\frac{\partial}{\partial x} dx + \frac{\partial}{\partial y} dy\right) \frac{\partial u}{\partial x} = \frac{\partial^2 u}{\partial x^2} dx + \frac{\partial^2 u}{\partial y^2} dy = rdx + sdy$$

$\Longrightarrow dp = rdx + sdy$, In this case let us take $\frac{\partial^2 u}{\partial x \partial y} = \frac{\partial^2 u}{\partial y \partial x}$

Similarly $dq = sdx + tdy$

Now the general linear equation in the second derivatives is of the form

$$Ar + Bs + Ct + Dp + Eq + Fu = G \longrightarrow \quad (1.2)$$

Note:

Equations (1.1) and (1.2) can be classified with respect to the **sign of the discriminant**.

Since (1.1) and (1.2) have discriminant

$$\Delta_s = B^2 - 4AC \longrightarrow \quad (1.3)$$

In the following manner:

a. If $\Delta_s < 0$, at the point in the x, y - plane, the equation is called **elliptic**.
b. If $\Delta_s > 0$, at the point in the x, y - plane, the equation is called **hyperbola**
c. If $\Delta_s = 0$, at the point in the x, y - plane, the equation is called **parabola.**

Note:

We can extend to the point (x, y) in n- dimensional space also in the same way we classified.

Our aim only considers HE that is parabolic equations.

1.2. Parabolic Equation:

As above discussion we easily conclude that, a partial differential equation in a region R. If $\Delta_s = B^2 - 4AC = 0$, at all points of the region, the initial value of the function u at time $t = 0$ and the normal derivative $\frac{\partial u(x,t)}{\partial t}$ on the boundary are the required boundary conditions. In problem of this type, the solution is not defined in a closed domain as in the case of Laplace's equation but is propagated in an open domain starting with prescribed conditions on an open boundary as shown in figure - 1 below:

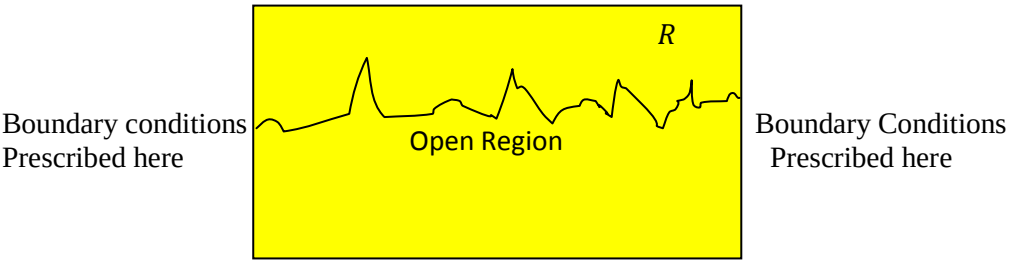

Figure 1

The one – dimensional heat – flow equation is$\frac{\partial u}{\partial t} = \alpha^2 \frac{\partial^2 u}{\partial x^2}$ it is an important equation of the parabolic type. We will see this equation how do develop to solve in the following different points of view.

2. FOURIER POINT OF VIEW [15]

In [15], Fourier considered variants of the following basic question. Let there be given an insulated, homogeneous rod of length$\pi$with initial temperature at each $x \in [0, \pi]$given by a function $f(x)$ (Fig. -2) Assume that the endpoints are held in temperature 0, and that the temperature of each cross – section is constant. The problem is to describe the temperature $u(x,t)$of the point$x$in the rod at time t. Fourier preserved the fundamental importance of this problem as follows:

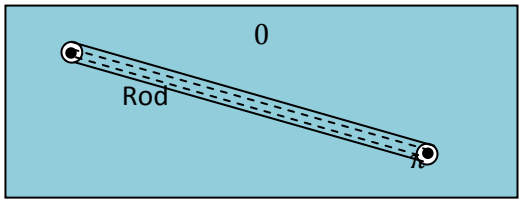

Figure 2

Primary causes are unknown to us; but are subject to simple and constant laws, which may be discovered by observation, the study of them being the object of natural philosophy.

Heat, like gravity, penetrates every substance of the universe, its rays occupying all parts of space. The object of our work is to set forth the mathematical laws which this elementary obey the theory of heat will form of the most important branches of general physics. Let us now describe the manner in which Fourier solved his problem. First, it is required to write a differential equation that$u$satisfies. We shall derive such an equation using three physical principles

a. The density of heat energy is proportional to the temperature$u$, hence the amount of heat energy in any interval$[a, b]$ of the rod is proportional to$\int_a^b u(x,t)\,dx$.
b. Newton's law of Cooling: The rate at which heat flows from a hot place to a cold one is proportional to the difference of the temperature. The infinite decimal version of this statement is that the rate of heat flow across a point$x$from left to right is some negative constant times $\partial_x u(x,t)$.

c. Conservation of Energy: Heat has no sources or sinks

Now (c) tells us that the only way that heat can enter or leave any initial portion $[a,b]$of the rod is through the end points. And (b) tells us exactly how this happens. Using (a), we may therefore write as

$\frac{d}{dt}\int_a^b u(x,t)\,dx = \eta^2[\partial_x u(b,t) - \partial_x u(a,t)]$ $where$ η^2, it is a positive constant of proportionality. We may rewrite this equation

$\int_a^b \partial_t u(x,t)\,dx = \eta^2 \int_a^b \partial_x^2 u(x,t)dx$, differentiating in b, we find that

$\partial_t u = \eta^2 \partial_x^2 u$, and that is the required HE.

An English biologist J. B. S. Haldane (1892 – 1964) had this remark about the one – dimensional HE "In scientific thought we adopt the simplest theory which will explain all the facts under consideration and enable us to predict new facts of the same kind". We can find such law as:

$$\alpha^2 \frac{\partial^2 w}{\partial x^2} = \frac{\partial w}{\partial t} \longrightarrow (2.1)$$

Assume that the constant of proportionality either $\eta^2 or\ a^2$equals 1. Fourier guessed the equation (1) has the solution of the form $u(x,t) = \alpha(x)\beta(t)$, substituting the guess in (1) yields

$$\alpha(x)\beta'(t) = \alpha''(x)\beta(t) or \frac{\alpha(x)}{\beta(t)} = \frac{\alpha''(x)}{\beta'(t)} \longrightarrow (2.2)$$

Since the left hand side of (2.2) is independent of$x$, and the right hand side of (2.2) is independent of$t$

$\Longrightarrow$There is a constant$K$, such that$\frac{\beta'(t)}{\beta(t)} = K = \frac{\alpha''(x)}{\alpha(x)}$ or $\begin{cases}\beta'(t) = K\beta(t)\\ \alpha''(x) = K\alpha(x)\end{cases}$

We conclude that $\beta(t) = Ce^{Kt}$, the nature of β, and hence of α, thus depend on the sign of$K$. But physical consideration tells us that the temperature will dissipate at times goes on, so we conclude that $K \le 0$

$\Longrightarrow \alpha(x) = \cos(\sqrt{-K}x)\ and \alpha(x) = \sin(\sqrt{-K}x)$, they are the solutions of the differential equation for α. The initial conditions $u(0,t) = u(\pi,t) = 0$

Since the ends of the rod are held at constant temperature 0, eliminate the first of these solutions and force $K = -j^2, and\ j$ as an integer. Thus Fourier found the solutions

$u_j(x,t) = e^{-j^2 t}\sin jx\,, for\ j \in \mathbb{N}$, of the heat equation

By linearity, any finite combination$\sum b_j\, e^{-j^2 t}\sin jx$, of these solutions is also a solution. This assertion can also extend to infinite linear combinations. Using the initial condition $u(x,0) = f(x)$, again raises the question of whether any $f(x)\ on\ [0,\pi]$, it can be written as a finite (infinite) linear combination of the function $\sin jx$

Fourier solution to this last question (of the function spanning essentially everything) is roughly as follows. Suppose that f, it is a function that is representable

$f(x) = \sum_j b_j \sin jx$ ⟶ (2.3)

Setting $x = 0$, it gives $f(0) = 0$

Differentiating both sides of (2.3), and setting $x = 0$, we have

$f'(0) = \sum_{j=1}^{\infty} jb_j$, successive differentiating k times and evaluate at 0, we get

$$f^{(k)}(0) = \begin{cases} \sum_{j=1}^{\infty} (-1)^{\frac{k}{2}} j^k b_j , & for\ k\ is\ odd < 0 \\ 0, & for\ k\ is\ even \geq 0 \end{cases}$$

Thus Fourier devised the system of infinitely many equations in infinitely many unknowns$\{b_j\}$. He proceeded to solve this system by truncating it to a $N \times N$ system (The first N equations restricted to the first$N$unknowns), solved that truncated system and let $N \longrightarrow \infty$. Suffice to say that Fourier's arguments contained many dubious steps [15, 17]

The upshot of Fourier's intricate and lengthy calculations was that

$b_j = \frac{2}{\pi} \int_0^{\pi} f(x) \sin jx\, dx$ ⟶ (2.4)

By modern standards, Fourier's reasoning was specious; for he began by assuming that$f$ possessed an expansion in terms of sine functions. The formula (2.4) hinges on that supposition, together with steps in which one compensated division by zero with a later division∞. Nevertheless, Fourier's methods give an actual procedure for endeavoring to expand any given f, in a series of sine functions, for example:

2.1. Example for Fourier's view of H E1:

"Suppose that the rod in the setup of the HE is first immersed in boiling water so that its temperature uniformly $90\,°C$. Then imagine that it is removed from the water at time $t = 0$ with its ends immediately put into ice so that these ends are kept at temperature $0°C$. Find the temperature $u = u(x,t)$ under these circumstances.

The initial temperature distribution is given by the constant function $f(x) = 90; for\ 0 < x < \pi$

The boundary conditions and other initial conditions are as usual. Thus our aim is to find the sine series expansion of this function f, note that $b_j = 0$, when$j$, it is even. For$j$, it is odd we calculate that

$b_j = \frac{2}{\pi} \int_0^{\pi} 90 \sin jx\, dx = -\frac{2\times 90}{\pi} \left[\frac{\cos jx}{j}\right]_0^{\pi} = \frac{360}{\pi j}$, as long as j, it is odd

$$\Longrightarrow f(x) = \frac{360}{\pi j}\left(e^{-t}\sin x + \frac{1}{3}e^{-9t}\sin 3x + \frac{1}{5}e^{-25t}\sin 5x + \cdots\right)$$

Now, referring to (2.3), and the preceding discussion from our general discussion of HE, we get

$$u(x,t) = \frac{360}{\pi j}\left(e^{-t}\sin x + \frac{1}{3}e^{-9t}\sin 3x + \frac{1}{5}e^{-25t}\sin 5x + \cdots\right)$$

2.2. Example for Fourier's view of H E 2

Find the Fourier series expansion of period 2π, for the function $f(x) = \sin \pi x in - \pi < x < \pi$

Solution:

Since $f(x) = \sin \pi x in - \pi < x < \pi$

$\Rightarrow f(x)$, it is an odd function then Fourier expansion of $f(x)$, contains only sine terms

$$\Rightarrow f(x) = \sum_{n=1}^{\infty} b_n \sin nx \, , Where \;\; b_n = \frac{2}{\pi}\int_0^{\pi} f(x) \sin nx \, dx$$

Since $\sin \pi x \sin nx = \frac{1}{2}[\cos(n-\pi)x - \cos(n+\pi)x]$

$$\Rightarrow b_n = \frac{2}{\pi}\int_0^{\pi} \frac{1}{2}[\cos(n-\pi)x - \cos(n+\pi)x]\, dx$$

$$\Rightarrow b_n = \frac{1}{\pi}\int_0^{\pi} [\cos(n-\pi)x - \cos(n+\pi)x]\, dx$$

$$\Rightarrow b_n = \frac{1}{\pi}\left[\frac{(\sin(n-\pi)x)}{n-\pi} - \frac{(\sin(n+\pi)x)}{n+\pi}\right]_0^{\pi}$$

Since $\sin 0 = 0 \Rightarrow b_n = \frac{1}{\pi}\left[\frac{(\sin(n-\pi)\pi)}{n-\pi} - \frac{(\sin(n+\pi)\pi)}{n+\pi}\right] = \frac{1}{\pi}\left[\frac{(\sin(n\pi-\pi^2))}{n-\pi} - \frac{(\sin(n\pi+\pi^2))}{n+\pi}\right]$

$$\Rightarrow b_n = \frac{1}{\pi}\left[\frac{(\sin n\pi cos\ \pi^2 - sin\pi^2 \cos n\pi)}{n-\pi} - \frac{(\sin n\pi cos\ \pi^2 + cos n\pi \sin \pi^2)}{n+\pi}\right]$$

Since $\sin n\pi = 0 \Rightarrow b_n = \frac{1}{\pi}\left[\frac{(-\sin \pi^2 \cos n\pi)}{n-\pi} - \frac{(sin\pi^2 \cos n\pi)}{n+\pi}\right]$

Since $\cos n\pi = (-1)^n \; and \; also \; sin\pi^2 \neq 0, because \; \pi$, it is neither 0 nor an integer

$$\Rightarrow b_n = \frac{1}{\pi}\left[\frac{((-sin\pi^2)(-1)^n)}{n-\pi} - \frac{(sin\pi^2)(-1)^n}{n+\pi}\right]$$

$$\Rightarrow b_n = \frac{(-1)^{n+1} sin\pi^2}{\pi}\left[\frac{1}{n-\pi} + \frac{1}{n+\pi}\right] = \frac{(-1)^{n+1} sin\pi^2}{\pi}\left[\frac{n+a+n-a}{n^2-\pi^2}\right]$$

$$= \frac{(-1)^{n+1} sin\pi^2}{\pi}\left[\frac{2n}{n^2-\pi^2}\right]$$

$$\Rightarrow b_n = (-1)^{n+1}\frac{2nsin\pi^2}{\pi(n^2-\pi^2)}$$

$$\Rightarrow f(x) = \sum_{n=1}^{\infty} b_n \sin nx = \sum_{n=1}^{\infty}\left\{(-1)^{n+1}\frac{2nsin\pi a}{\pi(n^2-\pi^2)}\right\}\sin nx$$

$$\Rightarrow f(x) = \frac{2\sin a\pi}{\pi}\sum_{n=1}^{\infty}\left\{\frac{n(-1)^{n+1}}{(n^2-\pi^2)}\right\}\sin nx$$

$$\Rightarrow \sin \pi x = \frac{2\sin \pi^2}{\pi}\sum_{n=1}^{\infty}\left\{\frac{n(-1)^{n+1}}{(n^2-\pi^2)}\right\}\sin nx$$

We conclude that Fourier point of view, we need lengthy calculations.

3. NUMERICAL VIEW OF FINITE DIFFERENCE METHOD (FDM)

Sometimes solving partial differential equation by ordinary methods is very difficult and laborious. So we go in for numerical methods of solving partial differential equations, the method of **finite differences** is most commonly used, in this method the partial differential coefficients is replaced by their finite difference approximations.

3.1. Solution of Parabolic Equation by FDM

We know that the one dimensional heat equation is

$$\frac{\partial u}{\partial t} = \alpha^2 \frac{\partial^2 u}{\partial^2 x} \quad where \;\; \alpha^2 = \frac{k}{e\rho}$$

Here e is the specific heat of the material, ρ is the density and k is the thermal conductivity.

The above heat equation can be written as

$$\frac{\partial^2 u}{\partial^2 x} = \frac{1}{\alpha^2}\frac{\partial u}{\partial t} \Longrightarrow u_{xx} = \frac{1}{\alpha^2}u_t = au_t, \qquad where \;\; a = \frac{1}{\alpha^2} \Longrightarrow u_{xx} - au_t = 0$$

Note that $A = 1, B = 0, C = 0, then B^2 - 4AC = 0, so \; that \;\; this \;\; equation \;\; is \;\; parabolic$

Let us solve by the method of finite differences the equation

$$u_{xx} = au_t \longrightarrow \quad (3.1)$$

With the boundary conditions,

$$\left.\begin{aligned} u(0,t) &= T_0 \\ u(l,t) &= T_1 \end{aligned}\right\} \longrightarrow \quad (3.2)$$

And the initial conditions

$$u(x,0) = f(x) \longrightarrow \quad (3.3)$$

We select spacing h for the variable x and spacing k in the time direction

We know that

$$u_{xx} = \frac{u_{i-1,j} - 2u_{i,j} + u_{i+1,j}}{h^2} and u_t = \frac{u_{i,j+1} - u_{i,j}}{k}$$

Equation (3.1) becomes

$\frac{u_{i-1,j} - 2u_{i,j} + u_{i+1,j}}{h^2} = a\left(\frac{u_{i,j+1} - u_{i,j}}{k}\right)$. That is,

$$u_{i,j+1} - u_{i,j} = \frac{k}{h^2 a}\left(u_{i-1,j} - 2u_{i,j} + u_{i+1,j}\right) = \lambda\left(u_{i-1,j} - 2u_{i,j} + u_{i+1,j}\right), \qquad where \;\; \lambda = \frac{k}{h^2 a}$$

It is called two levels formula because it is a relation between the function values at two levels

Or we have

$$u_{i,j+1} = \lambda u_{i+1,j} + (1 - 2\lambda)u_{i,j} + \lambda u_{i-1,j} \longrightarrow \quad (3.4)$$

It is called Explicit formula it valid if $0 < \lambda \leq \frac{1}{2}$

The boundary conditions (3.2) can be put in difference form as

$$\left.\begin{aligned} u_{0,j} &= T_0 \\ u_{n,j} &= T_1 \end{aligned}\right\} \; for\; j = 1, 2, \ldots \longrightarrow \quad (3.5)$$

Note that here$nh = l$

The initial condition (3.3) is

$$u_{i,0} = f(ih) \;\; for \;\; i = 1, 2, \ldots \longrightarrow \quad (3.6)$$

Equation (3.4) gives the value of

$$u \;\; at \;\; x = ih, \qquad at \;\; time \;\; t_j + k \;\; \text{interms} \; of \; values \; of \;\; u \;\; at \;\; x = (i-1)h, \qquad ih$$

$$and \; x = (i+1)h, \qquad at \; time \; t_j$$

Since $u(x, 0) = f(x), \;\; u \;\; is \;\; known \;\; at \;\; t = 0$

So the recurrence equation (3.4)

Allows the evaluation of $u \;\; at \;\; each \;\; pivotal \;\; point \;\; x_i \; at \;\; any \;\; time \; t_j$

Equation (3.4) becomes particularly simple, if for a given

$h, k \;\; is \; so \;\; chosen \;\; that \;\; the \;\; coefficient \; of \;\; u_{i,j} \;\; vanishes$. That is,

$1 - 2\lambda = 0 \Longrightarrow \lambda = \frac{1}{2}$, that is$\frac{k}{h^2 a} = \frac{1}{2} \;\; or \;\; k = \frac{h^2 a}{2}$. In this case (3.4) becomes

$$u_{i,j+1} = \frac{1}{2}\left[u_{i+1,j} + u_{i-1,j}\right] \longrightarrow \quad (3.7)$$

That is the value of $u \;\; at \;\; x = x_i$

$$at \;\; time \;\; t_{j+1} \;\; is \;\; equal \;\; to \;\; the \;\; average \; of \; the \;\; values \;\; of \;\; u \;\; at \;\; the$$

$surrounding \;\; points \;\; x_{i+1} \;\; and \;\; x_{i-1}$ at $the \;\; previous \;\; time \;\; t_j \;\; this \;\; is \;\; shown \;\; in \;\; figure$: 3

This method is called the **Bender - Schmidt** recurrence equation

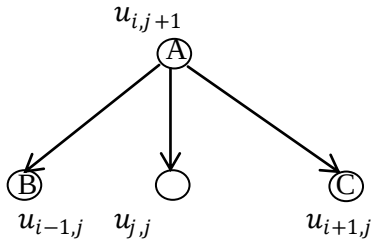

Figure 3 Schematic diagrams for Bender - Schmidt recurrence equation

$\Longrightarrow$The value of A =Average of the values at $B \; and \; C = \frac{1}{2}(B + C)$

Note that the solution obtained by using Bender Schmidt recurrence equation is stable when

$\lambda \leq \frac{1}{2}$, otherwise the solution is unstable and also the solution level at any point

$(i, j+1)$, on the $(j+1)th$ level it is expressed in terms of the solution value at the points

$(i-1, j), (i, j) \; and \; (i+1, j) \; on \; the \; jth \; level$, such a method is called explicit formula, so that Bender Schmidt recurrence equation is the explicit formula.

3.2. Example of Finite difference Bender – Schmidt Recurrence equation View:

Solve $u_{xx} = u_t$ subject to

$$u(0,t) = 0 = u(1,t) \; and \; u(x,0) = \sin \pi x, \;\; for \; 0 < x < 1$$

Since $h \; and \; k$, they are not given we will select them properly and use Bender – Schmidt method. Since

$k = \frac{a}{2}h^2 = \frac{1}{2}h^2 \; since \; a = 1$, and the range of $x \; is \; (0,1)$, and take $h = 0.2$

$$\Rightarrow k = \frac{(0.2)^2}{2} = 0.02$$

Since the formula is $u_{i,j+1} = \frac{1}{2}\{u_{i-1,j} + u_{i+1,j}\}$ by (3.7)

And also $u(0,0) = 0, u(0.2,0) = \sin\frac{\pi}{5} = 0.5875; \;\; u(0.4,0) = \sin\frac{2\pi}{5} = 0.9511;$

$u(0.6,0) = \sin\frac{3\pi}{5} = 0.9511; u(0.8,0) = \sin\frac{4\pi}{5} = 0.5875; \; and \;\; \sin(1,0) = 0,$

We can form the table as following manner:

$x - direction \; \rightarrow$

$t \downarrow direction$	i	0	1	2	3	4	5	
j		0	0.2	0.4	0.6	0.8	1.0	*Rows*
0	0	0	0.5875	0.9511	0.9511	0.5875	0	*Row* 1
1	0.02	0	0.4756	0.7695	0.7695	0.4756	0	*Row* 2
2	0.04	0	0.3848	0.6225	0.6225	0.3848	0	*Row* 3
3	0.06	0	0.3113	0.5306	0.5306	0.3113	0	*Row* 4
4	0.08	0	0.2511	0.4074	0.4074	0.2511	0	*Row* 5
5	0.1	0	0.2037	0.3296	0.3296	0.2037	0	*Row* 6

Explanation of the table working:

In the table by using formula (3.7) putting $j = 0$, we get $u_{i,1} = \frac{1}{2}\{u_{i-1,0} + u_{i+1,0}\}$ ⟶ (3.8)

Putting $i = 1$, in (3.8) we get, $u_{1,1} = \frac{1}{2}\{u_{0,0} + u_{2,0}\} = \left\{\frac{0+0.9511}{2}\right\} = 0.4756$

Putting $i = 2$, in (3.8) we get, $u_{2,1} = \frac{1}{2}\{u_{1,0} + u_{3,0}\} = \left\{\frac{0.5875+0.9511}{2}\right\} = 0.7695$

Putting $i = 3$, in (3.8) we get, $u_{3,1} = \frac{1}{2}\{u_{2,0} + u_{4,0}\} = \left\{\frac{0.9511+0.5875}{2}\right\} = 0.7695$

Putting $i = 4$, in (3.8) we get, $u_{4,1} = \frac{1}{2}\{u_{3,0} + u_{5,0}\} = \left\{\frac{0.9511+0}{2}\right\} = 0.4756$

In the above table second row is filled, similarly putting $j = 1,2,3,4 \; and \; 5$, the other rows are filled

Note:

Suppose assuming $h = k$=1,in this case $\lambda = \frac{k}{ah^2} = 1$, it violets the condition of for use of Explicit formula, and the solution is not stable, it is not a practical problem, since unstable solutions don't exist.

4. RADIAL BASIS FUNCTIONS VIEW (RBF) [1, 8, 9, 10]

Interpolation of a given set of points is an important problem especially in higher – dimensional domains. Even though polynomials is an important tool for interpolating a given set of points in one dimension, the use of these functions leads to difficulties in higher – dimensional domains. When we employed these functions, the arrangement of points in the domain of the problem which must have a certain form, however, this limits us when interpolation of a scattered set of points is needed. RBF are most efficient instruments for interpolating a scattered set of points which have been used in the last few years.

RBF methods have been introduced for interpolation of scattered data. Some well – known RBFs are listed below in table 1.

Table 1: Some Radial Basis Functions

Name of Function	Definition
Gaussian (GA)	$\psi(t) = \exp(-cr^2)$
Hardy Multi quadric(MQ)	$\psi(t) = \sqrt{r^2 + c^2}$
Inverse Multi quadric(IMQ)	$\psi(t) = \left(\sqrt{r^2 + c^2}\right)^{-1}$
Inverse quadric(IQ)	$\psi(t) = (r^2 + c^2)^{-1}$

Note:

In the above table $r = \|x^1 - x\|_2$, all functions are globally supported, infinitely differentiable, and depend on a free parameter c.

4.1. Property – Globally Supported Radial Functions[21]:

Let r, it is the Euclidean distance between$x^1 \in \mathbb{R}^d$, and any $x \in \mathbb{R}^d$, and then $r = \|x^1 - x\|_2$.

A radial function$\psi^1 = \psi(\|x^1 - x\|_2)$, depends only on the distance between $x \in \mathbb{R}^d$ and $x^1 \in \mathbb{R}^d$

This property gives us the radial functionψ^1, they are radially symmetric about x^1, and then, it is clear that the functions in above table: 1 is supported, infinitely differentiable, and depends on a free parameter c

Now, let us take$x_1, \dots, x_N$, it is a given set of distinct points in$\mathbb{R}^d$. The concept behind the use of RBFs is interpolation with the linear combination of RBFs of the same types as follows:

$F(x) = \sum_{i=1}^{n} \lambda_i \psi_i(x)$ ⟶ (4.1)

Where $\psi_i(x) = \psi(\|x - x_i\|)$ and λ_i, they are known scalars for $i = 1, \dots, N$.

The unknown scalars λ_i, they are chosen, so that $F(x_j) = f_j, for\ j = 1, \dots, N$, it results in the following linear system of equations:

$\boldsymbol{Ay} = \boldsymbol{f}$ ⟶ (4.2)

Where $\boldsymbol{A}_{i,j} = \psi_i(x_j); \boldsymbol{y} = [\lambda_1, \dots, \lambda_N]$ and $\boldsymbol{f} = [f_1, \dots, f_N]$

By the property of globally supported RBFs, we choose ψ global support, this method produces a dense matrix$\boldsymbol{A}$, by Schoenberg's theorem [18a]$\boldsymbol{A}$it is to be positive definite and therefore non – singular, for the distinct interpolation points for GA, IMQ and IQ, again by the Micchelli theorem [18a] matrix$\boldsymbol{A}$ it is invertible for distinct set of scattered set of points in the case of MQ. Even though matrix$\boldsymbol{A}$

It is non – singular, usually it is mostly ill – conditioned $\Longrightarrow$ the condition number of$\boldsymbol{A}$, it is

$\kappa_l(\boldsymbol{A}) = \|\boldsymbol{A}\|_l \|\boldsymbol{A}\|_l^{-1}, for\ l = 1, 2, \dots$ ⟶ (4.3)

It is a too large number$\Longrightarrow$a small perturbation in the initial data may produce large amount of perturbation in the solution $\Longrightarrow$ use more precise arithmetic's than the standard floating point arithmetic in this method's computation. For a fixed number of interpolation points, the condition number of $\boldsymbol{A}$ depends on the shape parameter c, support of the RBFs and the separation distance of interpolation points. Also, the condition number grows with N for fixed values of the shape of the parameter c. In practice, the shape parameter must be adjusted with the number of interpolating points in order to produce an interpolating matrix which is well conditioned enough to be inverted in finite precision arithmetic [19a]. Despite various research works which are done to develop algorithms for selecting the values of c, which produce most accurate interpolation (e.g. see [6, 19]), the optimal choice of shape parameter still as an open problem.

4.2. Method of Solution for One – Dimensional Problem:

Consider $Lu(x,t) = q(x,t), for\ (x,t) \in (a,b) \times (0,T]$ ⟶ (4.4)

Where$(a,b) \subset \mathbb{R}$, it is an open special interval and L it is a second – order linear parabolic operator, with the following initial and boundary conditions:

$u(x,0) = f(x)\ \ for\ x \in (a,b)$ ⟶ (4.5)

$u(a,t) = g_1(t)\ \ for\ \ t \in (0,T]$ ⟶ (4.6)

$u(b,t) = g_2(t)\ \ for\ \ t \in (0,T]$ ⟶ (4.7)

First, the domain$\Omega = (a,b) \times (0,T]$, it is decomposed as

$\Omega = \cup_{i=1}^{s} \Omega_i$ ⟶ (4.8)

Where$\Omega_i = [a,b] \times [(i-1)T_1, iT_1]$, for a sufficiently small value of$T_1 = \frac{T}{s}$ and s, it is a positive integer. We solve the problem in equations (4.4) to (4.7), in each sub domain Ω_i using the collocation technique, the sub domains are $\Omega_1, \Omega_2, \dots, \Omega_i, \dots \Omega_s$. For solving (4.4) to (4.7) in Ω_1, let

$\chi_1 = \{(x_i, t_i)\}_{i=1}^{N} \subset \Omega_1$ ⟶ (4.9)

It is a set of scattered nodes. Using the radial basis collocation method, the solution of the problem in$[a,b] \times [0,T_1]$, it is$\hat{u}_1 = \sum_{i=1}^{N} \alpha_{1i}\, \psi_i(x,t)$ ⟶ (4.10)

Where $\psi_i(x,t) = \psi(\|(x,t) - (x_i,t_i)\|)$, for a given radial function ψ and $\alpha_{1i}, for\ i = 1, \dots, N$

They are unknown. In fact RBFs are used for discretization of both time and space variables in $[a,b] \times [0,T_1]$. Now, by the collocation approach, impose the approximate solution$\hat{u}_1$ to satisfy the given differential equation with initial and boundary conditions at$(x_i, t_i), for\ i = 1, \dots, N$

⟹We have

$L\hat{u}_1(x_i,t_i) = q(x_i,t_i), for\ (x_i,t_i) \in (a,b) \times (0,T_1]$ ⟶ (4.11)

$\hat{u}_1(x_i,t_i) = f(x_i)\ \ for\ (x_i,t_i) \in [a,b] \times \{0\}$ ⟶ (4.12)

$\hat{u}_1(x_i,t_i) = g_1(t_i)\ \ for\ \ (x_i,t_i) \in \{a\} \times (0,T_1]$ ⟶ (4.13)

$\hat{u}_1(x_i,t_i) = g_2(t_i)\ \ for\ \ (x_i,t_i) \in \{b\} \times (0,T_1]$ ⟶ (4.14)

Those result in a linear system of equations

$\boldsymbol{A}\alpha_1 = \boldsymbol{b_1}$ ⟶ (4.15)

By solving this system of linear equations (4.15), unknown values are$\alpha_{1i}, for\ i = 1, \dots, N$,

They can be found. Generally, the obtained linear system is ill – conditioned. Here $\boldsymbol{A}$, it is $N \times N$ matrix. If T_1, it is chosen such that a small number of the collocation points provide an accurate approximation in$[a,b] \times [0,T_1]$;$\boldsymbol{A}$it will be a low – dimensional matrix

⟹ Finding Permutation Lower Upper triangular decomposition (PLU) of $\boldsymbol{A}$,

⟹ We have $\boldsymbol{PLU}(\boldsymbol{A})\alpha_1 = \boldsymbol{b_1}$ ⟶ (4.16)

Where$\boldsymbol{L} = [l_{ij}]$ and $\boldsymbol{U} = [u_{ij}]$, respectively with$l_{ii} = 1\ \ for\ i = 1, \dots, N$

Note that this factorization needs $\mathcal{O}(n^3)$, number of operations, and also$\boldsymbol{P} = \boldsymbol{P^{-1}}$, since$\boldsymbol{P}$, it is a permutation matrix

⟹ $\boldsymbol{LU}(\boldsymbol{A})\alpha_1 = \boldsymbol{Pb_1}$ ⟶ (4.17)

Again using the forward substitution with $\mathcal{O}(n^2)$, number of operations we solve

$\boldsymbol{Ly}_1 = \boldsymbol{Pb_1}$ ⟶ (4.18)

Then, findα_1, by solving$\boldsymbol{U}\alpha_1 = \boldsymbol{y}_1$, then using backward substitution it needs $\mathcal{O}(n^2)$, number of operations, consider the following problem in $\Omega_2 = [a,b] \times [T_1, 2T_1]$, it looked as

$Lu(x_i, t_i + T_1) = (x_i, t_i + T_1), for\ (x_i, t_i + T_1) \in (a,b) \times (T_1, 2T_1]$ ⟶ (4.19)

$u(x,0) = \hat{u}(x,T_1)\ \ for\ (x_i,t_i) \in [a,b] \times \{T_1\}$ ⟶ (4.20)

$u(a,t) = g_1(t)\ for\ (x,t) \in \{a\} \times (T_1, T]$ ⟶ (4.21)

$u(b,t) = g_2(t)\ for\ (x,t) \in \{b\} \times (T_1, T]$ ⟶ (4.22)

Set $\chi_2 = \{(x_i, t_i + T_1) | (x_i, t_i) \in \chi_1, for\ i = 1, \dots N\}$, as the set of collocation points in $[a,b] \times [T_1, 2T_1]$. Using the RBFs method the solution of the problemin$[a,b] \times [T_1, 2T_1]$ is

$\hat{u}_2 = \sum_{i=1}^{N} \alpha_{2i}\, \psi_i(x,t)$ ⟶ (4.23)

⟹We have

$Lu(x_i, t_i + T_1) = q(x_i, t_i + T_1), for\ (x_i, t_i + T_1) \in (a,b) \times (T_1, 2T_1]$ ⟶ (4.24)

$u(x_i, t_i + T_1) = \hat{u}(x_i, T_1)\ \ for\ (x_i, t_i) \in [a,b] \times \{T_1\}$ ⟶ (4.25)

$u(x_i, t_i + T_1) = g_1(t_i + T_1)\ \ for\ \ (x_i, t_i) \in \{a\} \times (T_1, 2T_1]$ ⟶ (4.26)

$u(x_i, t_i + T_1) = g_2(t_i + T_1) for\ \ (x_i, t_i) \in \{b\} \times (T_1, 2T_1]$ ⟶ (4.27)

Those result in a linear system of equations

$\boldsymbol{A}\alpha_2 = \boldsymbol{b_2}$ ⟶ (4.28)

Since$\hat{u}_2$, it is the linear combination of radial functions, and then same coefficient matrix be obtained. This property obtained from the fact that the value of a radial function is

$\psi^*(x,t) = \psi(\|(x,t) - (x^*, t^*)\|_2)$, depends only $\|(x,t) - (x^*, t^*)\|_2$, not the points$(x,t)\ and\ (x^*, t^*)$

⟹The unknown vector α_2, it is obtained from $\boldsymbol{L}\boldsymbol{y}_2 = \boldsymbol{P}\boldsymbol{b_2}\ and\ \boldsymbol{U}\alpha_2 = \boldsymbol{y}_2$ ⟶ (4.29)

In (4.29) we used only $\mathcal{O}(n^2)$, number of operations, we can precede remaining iterations, with choice of points in $\Omega_k = [a,b] \times [(k-1)T_1, kT_1]\ then\ \chi_k = (x_i, t_i + (k-1)T_1)\ \ for\ k = 1, \dots s$,

Finally we get$\boldsymbol{L}\boldsymbol{y}_k = \boldsymbol{P}\boldsymbol{b_k}\ and\ \boldsymbol{U}\alpha_k = \boldsymbol{y}_k\ for\ k = 2, 3, \dots$

It yields the approximate solution of the problemΩ_k ⟹the approximate solution is obtained in $[a,b] \times (0,T] = \cup_{i=1}^{s} \Omega_i$, only by LU - factorization of a low – dimensional matrix $\boldsymbol{A}$

In fact, in each sub – domain, the coefficient matrix of the resulted linear system which can be obtained from the discretization of the problem is same, therefore, in each sub – domain only the right hand side vector needs to be found.

4.3. Example for RBF's View in one dimensional case[20]:

Consider the heat equation $u_{xx} = u_t, for\ o < x < 1\ and\ 0 < t \leq T$ subject to

$$u(0,t) = 0 = u(1,t)\ for\ 0 < t \leq T\ and\ u(x,0) = \sin \pi x,\ \ for\ 0 < x < 1$$

We use Gaussian radial basis functions with $\Delta x = 0.1, c = 4, T_1 = 0.02$, and the set of equidistant collocation points for $N = 33$

First, this problem is solved in$[0,1] \times [0, 0.02]\ and\ \delta = 16$; floating point arithmetic's are used in this computation. The other approximations have been obtained only by forward and backward substitutions.

In this method, the solution of the problem is obtained in a closed form only by LU – decomposition of 33×33 matrix $\boldsymbol{A}$. In the following table – 2, some values of the shape parameter c, the condition number of matrix, the RMS error$E^2 = \sqrt{\frac{1}{M}\sum_{i=1}^{M}|u(x_i,t_i) - \hat{u}(x_i,t_i)|}$, and

The maximum error $E^{\infty} = \max_{1\leq i\leq M}|u(x_i,t_i) - \hat{u}(x_i,t_i)|$, they are listed for Gaussian RBF, and equidistant collocation points, where M, it is the number of collocation points $x \in [0,1]\ and\ t \in [0,1]$, for $\Delta x = 0.1\ \ and\ \Delta t = 0.01$, and also we conclude from the table – 2, the choice of the shape parameter has an auxiliary role in the stability of the problem. The dimension of matrix, it should be small sufficiently guarantee the stability of the solution of the resultant linear system.

Table 2: Shape of parameter values $c, E^2,\ E^{\infty}$, and corresponding condition number of matrix A

Shape parameter c	E^2	E^{∞}	Condition number of $\boldsymbol{A}$
0.5	90651×10^{36}	0.70000×10^{74}	0.19676×10^{20}
1	33136×10^{14}	0.20000×10^{30}	0.20434×10^{19}
2	0.50227	5.0001	0.93751×10^{18}
3	0.10196×10^{-1}	0.71266×10^{-3}	0.66616×10^{18}
4	0.10384×10^{-1}	0.74346×10^{-3}	0.40811×10^{18}
5	0.10400×10^{-1}	0.76029×10^{-3}	0.11529×10^{18}
6	0.10150×10^{-1}	0.75571×10^{-3}	0.14832×10^{17}
7	0.95715×10^{-2}	0.73241×10^{-3}	0.30267×10^{16}

5. View of Implicit Collocation Technique HE:

In this technique, consider the problem of finding $u(x,t)$, for the parabolic equation

$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial^2 x} + \emptyset(x,t)$ with $0 < x < 1\ \ and\ 0 < t \leq T$, and also with boundary conditions

$u(0,t) = g_0(t)\ \ for\ 0 < t \leq T\ \ and\ u(1,t) = g_1(t)\ \ for\ 0 < t \leq T$, again we need the nonlocal time weighting initial condition $u(x,0) = \sum_{j=1}^{N}\beta_j(x)\,u(x,T_j) + \varphi(x)$, with

$0 < T_1 < \cdots < T_N = T\ and\ 0 < x < 1\ \ where\ \emptyset, g_0, g_1, \beta_j, T_j\ \ and\ \varphi,$

They are known functions, while the function u, it is unknown.

The existence, uniqueness and continuous dependence of the solution of the above problem are shown in [6] with the following assumption that is the weights $\beta_j\ for\ j = 1, 2, \ldots N,$

It must satisfy the inequality

$\sum_{j=1}^{N}\|\beta_j(x)\|\, e^{-\pi^2 T_j} < 1, for\ 0 < x < 1$, here$\|.\|$, it is the maximum norm on$L^2(0,1)$.

In it is important that the maximum principle can be employed like the standard parabolic initial boundary value problems [4, 5 &7]. Our previous "Finite Difference Approximation view" is also developed for non standard initial or boundary condition for one – dimensional parabolic equations [2, 3, 12 & 13]. Before explain Collocation technique, we discussed again the method of evaluating $u(x,t)$, by Finite Difference Scheme.

5.1. Finite Difference Scheme (FDS):

First the domain$[0,1]\times[0,T]$, partitioned into an $M\times M$ mesh with the spatial steps size $h=\frac{1}{M}$, in x direction and the time step size $k=\frac{1}{N}$ respectively, grid points(x_i,t_j), they are defined by $x_i=ih\,,for\ i=0,1,\dots,M\ \ and\ t_j=jk,for\ k=0,1,\dots,N$, where M,N, they are integers.

The notations $u_i^j,\beta_{k,i},\varphi_i\ and\ \varphi_i^j$, they are used for the finite difference approximation of $u(x_i,t_j),\beta_k(x_i),\varphi(x_i)\ \ and\ \varphi(x_i,t_j)$, respectively, use direct simulation to the derivation of the One – dimensional equations $\frac{\partial u}{\partial t}=\frac{\partial^2 u}{\partial^2 x}+\emptyset(x,t)$ with $0<x<1\ \ and\ 0<t\le T$, with boundary conditions $u(0,t)=g_0(t)\ \ for\ 0<t\le T\ \ and\ u(1,t)=g_1(t)\ \ for\ 0<t\le T$

The explicit FDS leads to the following scheme for above equations that is

$$u_i^{j+1}=su_i^j+(1-2s)\left(u_{i-1}^j+su_{i+1}^j\right)+k\varphi_i^j \longrightarrow (5.1)$$

Using the initial condition $u(x,0)=\sum_{j=1}^{N}\beta_j(x)\,u(x,T_j)+\varphi(x)$, with $0<T_1<\cdots<T_N=T\ and\ 0<x<1\ \ where\ \emptyset\,,g_0\,,g_1,\beta_j,T_j\ \ and\ \varphi$, they are known functions, while the function u, it is unknown. We get

$$u_i^0=\sum_{j=1}^{N}\beta_{j,i}\,u_i^{N_j}+\varphi_i,\ \text{with}\ 1\le i\le M-1 \longrightarrow (5.2)$$

Using two given boundary conditions we get

$$u_i^0=0,u_M^j=0\ \ for\ 0\le j\le N \longrightarrow (5.3)$$

If a directed method is used to solve u_i^j we deal with a large non – linear system, but a simple iteration procedure be used because of the parabolic nature of the given problem.

Let us take$(u^0)^{(0)}=0$, it is the initial guess and then we get

$$\left(u_i^0\right)^{(l+1)}=\sum_{j=1}^{N}\beta_{j,i}\left(u_i^{N_j}\right)^{(l)}+\varphi_i,for\ i=1,2,\dots,M-1\ and\ l=0,1,2,\dots \longrightarrow (5.4)$$

$$where\ \left(u_i^j\right)^{(l)}$$

It is the finite solution of the forward Euler scheme with the initial data$\left(u_i^0\right)^{(l)}$

Note:

a. The range of stability for this procedure is $0\le s\le\frac{1}{2}$.[18]
b. This current explicit finite difference method for the numerical solution of the one – dimensional diffusion equation is the restriction of the size of the time step due to stability

requirements. These restrictions necessarily extremely small values of k, for most problems these are impractical methods, see the example of FDM.

5.2. The three – point – Implicit Backward Time, Centered Space (BTCS) Method:

It uses the formula over the time step$t_j \ to \ t_{j+1}$, it is

$$-su_{i-1}^{j+1} + (1+2s)u_i^{j+1} - su_{i+1}^{j+1} = u_i^j + k\emptyset_i^{j+1} \ for \ i = 1, 2, \dots M-1 \longrightarrow (5.5)$$

Note that the resulting system of linear algebraic equations is tri diagonal and it is solved by using Thomas algorithm. Again this method is uncondionally von Neumann stable [18], values of u_i^{j+1}on the boundaries $x = 0, 1$, they are provided by the given two boundary conditions. The main disadvantage of this implicit finite difference technique is the extensive amount of CPU times utilized in determining the numerical solution compared to the explicit method for the same selections values of $s \ and \ h$

5.3. Implicit Collocation Technique (ICT):

First the time – dependent partial differential equation is discredited in space, giving rise to a system of ordinary differential equations with unknown functions at each spatial grid points. The implicit collocation methods consist of approximating at each spatial grid point the solution by a polynomial that depends on time. Then determine the coefficients of all these polynomials. Depending on the given partial differential equation, we get a linear or non – linear system of equations, in this technique the system is linear [20], in which the unknowns are the coefficients. We can solve the resulting non – linear system by a direct or iterative technique. Once the coefficients of polynomials are determined, the approximated solution of the partial differential equation is computed on a given time interval that depends on the degree of the polynomial [16]

5.4. Procedure for ICT:

Let us take$u_i'(t) = \frac{u_{i-1}(t) - 2u_i(t) + u_{i+1}(t)}{h^2} + \emptyset_i(t) \ \ for \ i = 1, \dots, N-1 \longrightarrow$

(5.6)

Note that $u_i(0) = u(x_i, 0)$, and let$P_i(t)$, it is the rth degree of polynomial satisfying the system (5.9) (given last) at each time$t_j \ \ for \ j = 0, 1, \dots, r-1$

$$\Longrightarrow P_i(t_j) = a_0^i t_j^r + a_1^i t_j^{r-1} + \cdots + a_r^i \ \ for \ i = 1, \dots, N-1 \ \ and \ j = 1, \dots, r-1 \longrightarrow (5.7)$$

To solve$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial^2 x} + \emptyset(x,t)$ with $0 < x < 1 \ \ and \ 0 < t \leq T$, by the ICT the coefficients $a_0^i, \dots, a_{r-1}^i$ for $i = 1, \dots, N-1$, they should be determined.

The coefficient a_r^i, it can be determined from the initial condition that is

$$a_r^i = P_i(0) = u(x_i, 0) \ \ for \ i = 1, \dots, N-1 \longrightarrow (5.8)$$

Using (5.4) the coefficients $a_r^i, \ for \ r = 0, 1, \dots, r-1 \ and \ i = 1, \dots, N-1$, it can be obtained by the following procedure: This ICT developed by the combination of a finite difference scheme in

space and a collocation technique in time and also it is employed is a second – order approximation in space and an approximation of order r in time. With implicit collocation method, the solution of the one – dimensional diffusion equation is approximated by polynomials. For a given time step, the time interval where the solution is computed depends on the degree of the polynomials. The higher, this degree is the higher order of the method is and the longer the time interval is and therefore it is tempting to choose polynomials of high degree, in order to approximate the solution is relatively long time interval by solving one linear system. To solve given parabolic equation by collocation technique, a linear system which will be introduced [20]

First we introduce some notations, they are

$$X = \left[[a_0^1, \dots, a_{r-1}^1], \dots, [a_0^{N-1}, \dots, a_{r-1}^{N-1}]\right]^T \text{ and } B = \left[[a_r^1, \dots, a_r^1], \dots, [a_r^{N-1}, \dots, a_r^{N-1}]\right]^T$$

$$P = \left[[P_1(t_0), \dots, P_1(t_{r-1})], \dots, [P_{N-1}(t_0), \dots, P_{N-1}(t_{r-1})]\right]^T$$

$$Q = \left[[u_0(t_0), \dots, u_o(t_{r-1})], \dots, [u_N(t_0), \dots, u_N(t_{r-1})]\right]^T$$

$$S = [[u_0(t_0) - 2a_r^1 + a_r^2, \dots, u_o(t_{r-1}) - 2a_r^1 + a_r^2]^T, \dots, [a_r^1 - 2a_r^2 + a_r^3, \dots, a_r^1 - 2a_r^2 + a_r^3]^T, \dots, [a_r^{N-2} - 2a_r^{N-1} + u_N(t_0), \dots, a_r^{N-2} - 2a_r^{N-1} + u_N(t_{r-1})]^T]^T$$

$$P' = [[P_1'(t_0), \dots, P_1'(t_{r-1})]^T, \dots, [P_{N-1}'(t_0), \dots, P_{N-1}'(t_{r-1})]^T]^T$$

$$\alpha = \begin{bmatrix} t_0^r & t_0^{r-1} & \cdots & t_0 \\ t_1^r & t_1^{r-1} & \cdots & t_1 \\ \vdots & \vdots & \ddots & \vdots \\ t_{r-1}^r & t_{r-1}^{r-1} & \cdots & t_{r-1} \end{bmatrix} \text{ and } A = Diagonal\ (\alpha, \dots, \alpha),$$

Where A, it is a block – diagonal matrix of order $r(N-1)$, and then we get

$$\alpha' = \begin{bmatrix} rt_0^{r-1} & (r-1)t_0^{r-2} & \cdots & 2t_0 & 1 \\ rt_1^{r-1} & (r-1)t_1^{r-2} & \cdots & 2t_1 & 1 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ rt_{r-1}^{r-1} & (r-1)t_{r-1}^{r-2} & \cdots & 2t_{r-1} & 1 \end{bmatrix} \text{ and } T = \begin{bmatrix} -2I_r & I_r & 0 & \cdots & 0 \\ I_r & -2I_r & I_r & \ddots & \vdots \\ 0 & \ddots & \ddots & \ddots & 0 \\ \vdots & \ddots & I_r & -2I_r & I_r \\ 0 & 0 & 0 & I_r & -2I_r \end{bmatrix}$$

WhereI_r, it is the identity matrix of order r, and also T, it is a block – diagonal matrix of order $r(N-1$, and then we get $M=DC0:0CD\ddots\ddots\cdots0C\ddots C0\cdots\ddots\ddots DC0:0CD$

$$Where\ C_{i,j} = -t_{i-1}^{r+1-j} \text{ and } D_{i,j} = t_{i-1}^{r-j}(h^2(r+1-j) + 2t_{i-1})\ for\ i,j = 1,2,\dots,r$$

We can simply write as

$S = TB + Q$ and $M = h^2A' - TA$, and also the vector X, it is the solution of the linear system

$$MX = S \longrightarrow (5.9)$$

Note:

a. If$|D| = 0$, if and only if two valuest$_{i-1}$ and t_{j-1}, they are identical. As the points t_{i-1} $for\ i = 1, \dots, r$, they are all distinct.

b. If$|D| \neq 0$, and then the linear system $MX = S$, it has a unique solution.

Example for ICT:[16]

Considerthe one – dimensional equations$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial^2 x} + \emptyset(x,t)$ with $0 < x < 1$ and $0 < t \leq T$, with boundary conditions $u(0,t) = g_0(t)$ $for\ 0 < t \leq T$ $and\ u(1,t) = g_1(t)$ $for\ 0 < t \leq T\ with\ N = 2$

To solve$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial^2 x} + \emptyset(x,t)$, with boundary conditions and the non standard initial conditions

$u(x,0) = \sum_{j=1}^{N} \beta_j(x)\, u(x,T_j) + \varphi(x)$, in this we have

$N = 2$ and $+\emptyset(x,t) = (-1+\pi^2)\sin\pi x\, e^{-t}$ ⟶ (5.10)

$u(x,0) = u(x,T_1) - u(x,T_2) + \varphi(x)$ ⟶ (5.11)

$\varphi(x) = \sin\pi x\,(1 - e^{T_1} + e^{T_2})$ ⟶ (5.12)

$g_0(t) = 0 = g_1(t)$ ⟶ (5.13)

For which the exact solution is $u(x,t) = e^{-t}\sin\pi x$

And also $0 < T_1 < T_2 = T;\ \beta_1 = 1$ and $\beta_1 = -1$, the results of the error inL^1norm, with $T_1 = 0.5, T_2 = 1$, computed for various values of h and k using this schemes report with $u(x,0)$, in (5.10) in a non – standard form and found by (5.4) are obtained. In order to keep the accuracy the stopping criteria ε of the iteration tolerance is chosen by $\varepsilon = 0.5(h^2 + k)$, in all above computations. That is let the initial guess

$u^0 = 0$, and if$\left|(u^0)^{(l+1)} - (u^0)^{(l)}\right| \leq \varepsilon, for\ some\ l, then\ \left(u_i^0\right)^{(l+1)}$ and $\left(u_i^N\right)^{(l+1)}$, they will be accepted as the numerical initial value and final value respectively, and the computation will be terminated: The results obtained by various values of T_i showed that what the initial value we start with $u(x,t)$ approaches the steady state quickly as t increases, thus $u(x,t)$, it can be calculated accurately for larger t, which in turn gives a good next step initial updated data.

6. CRANK – NICOLSON SCHMIDT VIEW[11]

For the parabolic equation

$$u_{xx} = u_t$$

$u(0,t) = T_0$ And the initial conditions

$$u(l,t) = T_1$$

$$u(x,0) = f(x), for\ 0 < x < l$$

6.1. Procedure:

The given equation is $u_{xx} = au_t$ ⟶ (6.1)

At the point $u_{j,j}$ we have the following finite difference approximation for u_{xx} on the j^{th} row

$$u_{xx} \approx \frac{u_{i-1,j} - 2u_{i,j} + u_{i+1,j}}{h^2}$$

Similarly at the point $u_{j,j+1}$, we have

$$u_{xx} \approx \frac{u_{i-1,j+1} - 2u_{i,j+1} + u_{i+1,j+1}}{h^2}$$

Averaging these two approximations, we obtain

$u_{xx} \approx \frac{u_{i-1,j+1} - 2u_{i,j+1} + u_{i+1,j+1} + u_{i-1,j} - 2u_{i,j} + u_{i+1,j}}{2h^2}$ ⟶ (6.2)

For u_t, we use forward difference approximation

$u_t = \frac{u_{i,j+1} - u_{i,j}}{k}$ ⟶ (6.3)

Substitute (6.2) and (6.3) in (6.1) we get

$$\frac{u_{i-1,j+1} - 2u_{i,j+1} + u_{i+1,j+1} + u_{i-1,j} - 2u_{i,j} + u_{i+1,j}}{2h^2} = \frac{a(u_{i,j+1} - u_{i,j})}{k}$$

$$\Rightarrow \frac{k}{2h^2 a}(u_{i-1,j+1} - 2u_{i,j+1} + u_{i+1,j+1} + u_{i-1,j} - 2u_{i,j} + u_{i+1,j}) = (u_{i,j+1} - u_{i,j})$$

Setting $\frac{k}{2h^2 a} = \lambda$, we have

$$\frac{1}{2}\lambda u_{i+1,j+1} - \lambda u_{i,j+1} + \frac{1}{2}\lambda u_{i-1,j+1} - u_{i,j+1} = \lambda u_{i,j} - \frac{1}{2}\lambda u_{i+1,j} - \frac{1}{2}\lambda u_{i-1,j} - u_{i,j}$$

$\Rightarrow \frac{1}{2}\lambda u_{i-1,j+1} - (\lambda + 1)u_{i,j+1} + \frac{1}{2}\lambda u_{i+1,j+1} = -\frac{1}{2}\lambda u_{i-1,j} + (\lambda - 1)u_{i,j} - \frac{1}{2}\lambda u_{i+1,j}$ ⟶ (6.4)

This is called the general form of Crank Nicolson difference scheme. This equation (6.4) also called the implicit formula, as it does not give the value of $u \ at \ t = t_{j+1}$ directly in terms of the values of $u \ at \ t = t_j$. Though λ it can take any value, we take $\lambda = 1$ in order to simplify the numerical work involved. When $\lambda = 1$, the Crank Nicolson's difference equation takes the simplest form, and then Equation (6.4) becomes,

$$\Rightarrow \frac{1}{2}u_{i-1,j+1} - (1 + 1)u_{i,j+1} + \frac{1}{2}u_{i+1,j+1} = -\frac{1}{2}u_{i-1,j} - \frac{1}{2}u_{i+1,j}$$

$\Rightarrow u_{i-1,j+1} - 4u_{i,j+1} + u_{i+1,j+1} = -u_{i-1,j} - u_{i+1,j}$ Or

$-u_{i-1,j+1} + 4u_{i,j+1} - u_{i+1,j+1} = u_{i-1,j} + u_{i+1,j}$ ⟶ (6.5)

As far as possible, we should try to make use of equation (6.5), by proper choice of

$h \ \ either \ 'and', 'or' k, \ sothat \ \lambda = \frac{k}{2h^2 a} = 1$

Note: we have six points in Crank Nicolson's method in (6.5), that is

$$u_{i-1,j}, u_{i,j}, u_{i+1,j}, and \ u_{i-1,j+1}, u_{i,j+1} \ and \ u_{i+1,j+1}$$

These six points are shown in figure (4):

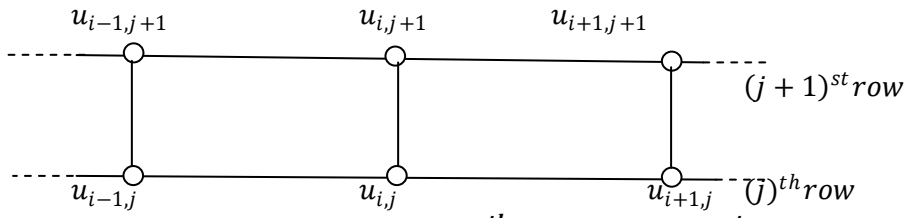

Figure: 4 Crane -Nicolson's Scheme in$(j)^{th}row\ and(j+1)^{st}row$

Note:

A convenient choice of $\lambda = 1$, it makes the Crane Nicolson's scheme becomes simple, that is $k = ah^2$

Then we get modified form of Crane Nicolson's Scheme is

$$u_{i,j+1} = \frac{1}{4}\left[u_{i-1,j+1} + u_{i+1,j+1} + u_{i-1,j} + u_{i+1,j}\right] \longrightarrow (6.6)$$

So that we will use this simplified formula subject to $k = ah^2$, and also Crane Nicolson's Scheme as shown in figure (5):

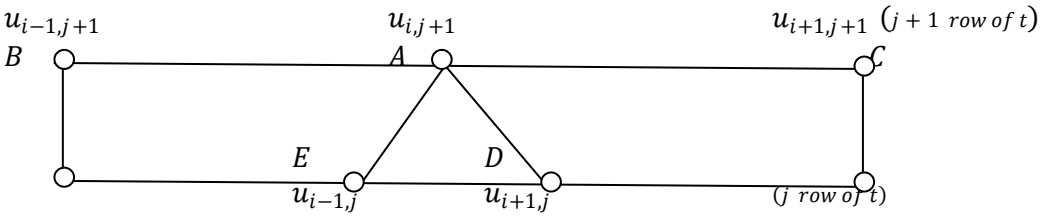

Figure: 5 Crane Nicolson's Scheme

$\Longrightarrow$The value of A =Average of the values at $B, C, D\ and\ E$

Note:

1. The Crank Nicolson's scheme converges for all finite values of λ.
2. It is an implicit scheme because,

   On the left side of system of linear equations (6.6) we have four unknowns and on the right sides all the four quantities are known. Equation (6.6) which is an implicit scheme:
3. If there are N internal mesh points on each row, then the CNS formula gives N simultaneous equations for the N unknowns in terms of the given boundary values.
4. Similarly, the internal mesh points on all rows can be calculated.
5. If there are N internal mesh points on each row, then formula (6.6) given N simultaneous equations for the N unknowns in terms of the given boundary values. Similarly, the internal mesh points on all rows can be calculated.
6. CNS likes BTCS and unconditional stable, even though BTCS and CNS have similar procedure, but CNS scheme has a truncation error of $\mathcal{O}(\Delta x^2) + \mathcal{O}(\Delta t^2)$, that is temporal truncation error is significantly smaller than the temporal error of BTCS scheme.

6.2. Example for CNS View:

Apply Crank – Nicolson method with$h = 0.2 \ and \ \lambda = 1$ and find $u(x,t)$, in the rod by considering two time steps of the heat equation$u_{xx} = u_t$ satisfying the conditions

$u(x,0) = \sin \pi x \ and \ u(0,t) = 0 = u(t,t)$, and prepare table form as shown below

$x - direction \ \rightarrow$

t ↓	i / j		0	1	2	3	4	5	
			0	0.2	0.4	0.6	0.8	1.0	
			0	0.588	0.951	0.951	0.588	0	*Rows*
	0	0	0	u_1	u_2	u_3	u_4	0	
			0	0.399	0.646	0.646	0.399	0	*Row* 1
	t_1	0.2	0	u_5	u_6	u_7	u_8	0	
			0	0.271	0.439	0.439	0.2710		*Row* 2

6.3. Explanation of above table working:

First in the table,since $u(x,0) = \sin \pi x$

$$\Longrightarrow u(0,0) = 0;\ u(0.2,0) = \sin\frac{\pi}{5} = 0.5878 \approx 0.588;\ u(0.4,0) = \sin\frac{2\pi}{5} = 0.9511 \approx 0.951$$

$$\Longrightarrow u(0.6,0) = \sin\frac{3\pi}{5} = 0.9511 \approx 0.951; u(0.8,0) = \sin\frac{4\pi}{5} = 0.578 \approx 0.588 \ and$$

$$u(1,0) = \sin\frac{5\pi}{5} = 0$$

Filling the place of in the value of$x - direction$

Secondly, in the table we need

$u_1 = \frac{0+0.951+0+u_2}{4}; u_2 = \frac{u_1+0.558+0.951+u_3}{4};\ u_3 = \frac{u_2+0.951+0.588+u_4}{4}\ ; u_4 = \frac{u_3+0.951+0+0}{4}$, by (6.6)

We simply solve in terms$u_1, u_2, u_3 \ and \ u_4$, we get the following system of linear equations

$$u_1 = 0.23775 + \frac{1}{4}u_2 \Longrightarrow 4u_1 - u_2 = 0.951$$

$$u_2 = \frac{1}{4}u_1 + 0.038475 + \frac{1}{4}u_3 \Longrightarrow -u_1 + 4u_2 - u_3 = 0.1539$$

$$u_3 = \frac{1}{4}u_2 + 0.038475 + \frac{1}{4}u_4 \Longrightarrow -u_2 + 4u_3 - u_4 = 1.539$$

$$u_4 = \frac{1}{4}u_3 + 0.23775 \Longrightarrow -u_3 + 4u_4 = 0.951$$

So that we get the following system of linear equations in terms ofu_1, u_2, u_3, u_4, they are

$$\left.\begin{aligned} 4u_1 - u_2 &= 0.951 \\ -u_1 + 4u_2 - u_3 &= 0.1539 \\ -u_2 + 4u_3 - u_4 &= 1.1539 \\ -u_3 + 4u_4 &= 0.951 \end{aligned}\right\} \longrightarrow \quad (6.7)$$

Solve this system (6.7) (Note that this system has coefficient matrix is in tri – diagonal matrix form) by any method (like as elimination, substitution and others) then we found that

$u_1 = 0.399, u_2 = 0.646, u_3 = 0.646$ *and* $u_4 = 0.399$, we can filled the first row elements in the above table, similarly we will fill second row elements u_5, u_6, u_7 *and* u_8.

7. CONCLUSION

By the Fourier point of view we will get the result after long calculations, and its convergence is depends upon continuity, boundedness and other conditions[16, 17], by FTM – Bender Schmidt method, we will get the results after some more steps for the same given example, but explicit scheme has convergence only limited in $0 < \lambda \leq \frac{1}{2}$, RBF also give the results near to exact solution but we need number of $\mathcal{O}(n^3) + 2\mathcal{O}(n^2)$operations to get the solutions. ICT also needs more calculations and number of iterations to reach near to exact solution and also the accuracy of numerical calculations not only depend on the step sizes but also on the two parameters β_1 *and* β_2 But CNS, needs few steps by solving system of linear equations depend up on number of unknownsu_i, $for\ i = 1, 2, \ldots$ to find the solutions of one – dimensional heat equations to reach near to exact solution needsonly choosing λ, properly,and also no need more calculations with short time, finally it convergence for all values of λ, and truncation error also significantly smaller than other approaches so that, from this article we conclude that CNS is still as a major role to solve one – dimensional heat equation.

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