HOMOGENEOUS BALANCE METHOD FOR FORNBERG-WHITHAM (FM) EQUATION

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Abstract: *In this article, the traveling wave solutions of the Fornberg-Whitham equation are studied by using the approach of homogeneous balance method. With this method, the Fornberg-Whitham equation is reduced to the nonlinear ordinary differential equation and then the different types of exact solutions are derived based on the solutions of the Riccati equation. The applied method is more powerful and will be used in further works to establish more entirely new solutions for other kinds of nonlinear partial differential equations arising in mathematical physics.*

Keywords: *Homogeneous balance method, Fornberg-Whitham (FM) equation.*

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1 INTRODUCTION

The investigation of traveling wave solutions to nonlinear partial differential equations plays an important role in study of nonlinear phenomena. Nonlinear waves appear in a wide variety of scientific applications such as fluid dynamics, plasma physics, optical fibers etc. Several powerful methods have been proposed to obtain exact solutions of nonlinear equations such as, Bäcklund transformation method(Fan 2002; Wang 2001), the improved tanh-method (Yasar 2010), homotopy analysis method (El-Wakil1 2010), homotopy perturbation method (He 1999,2000; Gupta and Singh 2011; Singh; et al. 2010; Aminikhaha and Salahi 2010; Yildrim 2010; Biazar, Aminikhah 2009; Mohyud-Din et al. 2010; Biazar, J., Eslami 2011). In recent year, the homogeneous balance HB Method has been widely used to derive the nonlinear transformation and exact solutions (Fan 1998, 2000; Fan and Zhang 1998; Zhao and Tang 2002; Wand 1996; Wang and Zhou, Abdel Rady and Osman 2010)

In this paper, we consider the Fornberg-Whitham equation in the following form (Gupta and Singh 2011):

$$u_t - u_{xxt} + u_x - uu_{xxx} + uu_x - 3u_x u_{xx}, \qquad x \in R,\ t > 0, \tag{1}$$

where $u(x,t)$ is unknown function; t is the time; x is the single spatial coordinate.

2 TRAVELING WAVE SOLUTIONS TO FORNBERG WHITHAM EQUATION

In this Section, we consider the HBM introduced in (Abdel Rady and Osman 2010). To obtain the travelling wave solution of equation (1).

Let us consider the solution of equation (1) in the following form:

$$u(x,t) = \sum_{i=0}^{n} \alpha_i \psi_i = u(\eta), \tag{2}$$

where α_i is parameters, $\eta = kx + \lambda t + \beta$ in which $k, \lambda,$ and β are constants and ψ is solution of Riccati equation

$$\psi' = A + B\psi(\eta) + C\psi^2(\eta), \tag{3}$$

where prime denotes ordinary differentiation with respect to η; A, B, and C are constants.

Substituting Eq. (2) in (1), we obtain

$$(\lambda + k)k^2 u''' + 3k^3 u'u'' - (1+u)ku' - (\lambda + k^3)u = 0, \tag{4}$$

We may choose for $n = 2$ of equation (1):

$$u = a_0 + a_1\psi + a_2\psi^2, \tag{5}$$

Substituting equation (5) in (4) along with (3) and using Mathematica-6 yields a system of equations w. r. t ψ^i. Setting the coefficients of ψ^i (($i = 0, 1, ...6$) in the obtained system of equations to zero, we can get the following set of algebraic polynomials with the respect unknown α_0, α_1 and α_2 namely:

$$A\lambda + Bka_0 + kABa_0 - 6BCk^3a_0^3 - 3B^2k^3a_0^2a_1 - 6ACk^3a_0^2a_1 - 6Ck^2\lambda a_0^2a_1 - ABk^3a_0a_1^2$$
$$-Bk^2\lambda a_0a_1^2 - 2ABk^3a_0a_2 - 2Bk^2\lambda a_0^2a_2 = 0, \tag{6}$$

$$B\lambda + B^2ka_0 + 2Cka_0 + 2ACka_0 - 12C^2k^3a_0^3 + Bka_1 + ABka_1 - 36BCk^3a_0^2a_1 - 7B^2k^3a_0a_1^2$$
$$-14ACk^3a_0a_1^2 - 14Ck^2\lambda a_0a_1^2 - ABk^3a_1^3 - Bk^2\lambda a_1^3 - 8B^2k^3a_0^2a_2 - 16ACk^3a_0^2a_2 - 16Ck^2\lambda a_0^2a_2$$
$$-8ABk^3a_0a_1a_2 - 8Bk^3\lambda a_0a_1a_2 = 0, \tag{7}$$

$$C\lambda + 3BCka_0 + B^2ka_1 + 2Cka_1 + 2ACka_1 - 54C^2k^3a_0^2a_1 - 57BCk^3a_0a_1^2 - 4B^2k^3a_1^3 - 8ACk^3a_1^3$$
$$-8Ck^2\lambda a_1^3 + Bka_2 + ABka_2 - 60BCk^3a_0^2a_2 - 26B^2k^3a_0a_1a_2 - 52ACk^3a_0a_1a_2 - 52Ck^2\lambda a_0a_1a_2$$
$$-7ABk^3a_1^2a_2 - 7Bk^2\lambda a_1^2a_2 - 8ABk^3a_0a_2^2 - 8Bk^2\lambda a_0a_2^2 = 0, \tag{8}$$

$$2C^2ka_0 + 3BCka_1 - 74C^2k^3a_0a_1^2 - 27BCk^3a_1^3 + B^2ka_2 + 2Cka_2 + 2ACka_2 - 76C^2k^3a_0^2a_2$$
$$-168BCk^3a_0a_1a_2 - 19B^2k^3a_1^2a_2 - 38ACk^3a_1^2a_2 - 38Ck^2\lambda a_1^2a_2 - 20B^2k^3a_0a_2^2 - 40ACk^3a_0a_2^2$$
$$-40Ck^2\lambda a_0a_2^2 - 12ABk^3a_1a_2^2 - 12Bk^2\lambda a_1a_2^2 = 0, \tag{9}$$

$$2C^2ka_1 - 32C^2k^3a_1^3 + 3BCka_2 - 196C^2k^3a_0a_1a_2 - 111BCk^3a_1^2a_2 - 114BCk^3a_0a_2^2 - 27B^2k^3a_1a_2^2$$
$$-54ACk^3a_1a_2^2 - 54Ck^2\lambda a_1a_2^2 - 6ABk^3a_2^3 - 6Bk^2\lambda a_2^3 = 0, \tag{10}$$

$$2C^2ka_2 - 122C^2k^3a_1^2a_2 - 124C^2k^3a_0a_2^2 - 144BCk^3a_1a_2^2 - 12B^2k^3a_2^3$$
$$-24ACk^3a_2^3 - 24Ck^2\lambda a_2^3 = 0, \tag{11}$$

$$-150C^2k^3a_1a_2^2 - 60BCk^3a_2^3 = 0, \tag{12}$$

The solution of the algebraic system of equations given by (6)-(12) with the help of Mathematica-6, can be determined as follows:

$$a_0 = \pm\frac{\sqrt{-400B^4 + 1000B^2C\left(A + \frac{\lambda}{k}\right) - 625A^2\left(C^2 + \frac{\lambda^2}{k^2}\right) - \frac{1250AC^2\lambda}{k}}}{C\sqrt{k}\sqrt{-27456B^2k + 49800C(Ak + \lambda)}}, \tag{13a}$$

$$a_1 = \frac{155BCk^2a_0 \pm \sqrt{-326B^4k^2 + 300B^2Ck(Ak + \lambda) + 24025B^2C^2k^4a_0^2}}{-163B^2k^2 + 150Ck(Ak + \lambda)}, \tag{13b}$$

$$a_2 = \frac{5C}{2}\left[\frac{-155Ck^2a_0 \mp \sqrt{-326B^2k^2 + 300Ck(Ak+\lambda) + 24025C^2k^4a_0^2}}{-163B^2k^2 + 150Ck(Ak+\lambda)}\right]. \tag{13c}$$

It is noted that the Riccati equation (3) can be solved using homogeneous balance method as follows

Case I. Let $\psi = \sum_{i=0}^{m} \sigma_i \tanh^i \eta$. Balancing ψ' with ψ^2 leads to

$$\psi = \sigma_0 + \sigma_1 \tanh \eta . \tag{14}$$

Substituting (14) in (3), we obtain the following solution of (3)

$$\psi = -\frac{1}{2A}\left(B + 2\tanh \eta\right), AC + 1 = B^2/4 . \tag{15}$$

Substituting equations (13a)-(13c) and (14) into (5) and (2), we have the following traveling wave solution of Eq. (1)

$$u(x,t) = \frac{1}{A^2}\left[A^2a_0 - \frac{B(2Aa_1 - Ba_2)}{4}\right] + \frac{1}{A^2}\left[Ba_2 - a_1A\right]\tanh\eta + \frac{a_2}{A^2}\tanh^2\eta . \tag{16}$$

Similarly, let $\psi = \sum_{i=0}^{m} B_i \coth^i \eta$, then we obtain the following new traveling wave solution of equation (1)

$$u(x,t) = \frac{1}{A^2}\left[A^2a_0 - \frac{B(2Aa_1 - Ba_2)}{4}\right] + \frac{1}{A^2}\left[Ba_2 - a_1A\right]\coth\eta + \frac{a_2}{A^2}\coth^2\eta . \tag{17}$$

Case-II: When we choose $A = -\frac{1}{2}$, $B = 0$ and $C = \frac{1}{2}$ then we can found the value of coefficients.

$$a_0 = \frac{\pm i(k-\lambda)}{4k\sqrt{249k\left(\lambda - \frac{k}{2}\right)}},\ a_1 = 0,\ a_2 = \frac{\dfrac{\pm 31i(k-\lambda)}{4\sqrt{249k\left(\lambda - \frac{k}{2}\right)}} \mp \sqrt{6\lambda k - 3 - \dfrac{961}{7968}\dfrac{(k-\lambda)^2}{k\left(2\lambda - k\right)}}}{2\left(2\lambda - k\right)}$$

Therefore, the solution can be obtained as

$$u(x,t)=\frac{\pm i(k-\lambda)}{4k\sqrt{249k\left(\lambda-\frac{k}{2}\right)}}+\frac{\left(\frac{\pm 31i(k-\lambda)}{4\sqrt{249k\left(\lambda-\frac{k}{2}\right)}}\mp\sqrt{6\lambda k-3-\frac{961}{7968}\frac{(k-\lambda)^2}{k(2\lambda-k)}}\right)}{2(2\lambda-k)}\tanh^2(kx+\lambda t+d) \quad (18)$$

$$u(x,t)=\frac{\pm i(k-\lambda)}{4k\sqrt{249k\left(\lambda-\frac{k}{2}\right)}}+\frac{\left(\frac{\pm 31i(k-\lambda)}{4\sqrt{249k\left(\lambda-\frac{k}{2}\right)}}\mp\sqrt{6\lambda k-3-\frac{961}{7968}\frac{(k-\lambda)^2}{k(2\lambda-k)}}\right)}{2(2\lambda-k)}\coth^2(kx+\lambda t+d) \quad (19)$$

Case-III: When we choose $A=-\frac{1}{2}, B=0, C=\frac{1}{2}$ and $k=\lambda$ then we can found coefficients $a_0=0$, $a_1=0$ and $a_2=\frac{\sqrt{3(2k^2-1)}}{2k}$. Therefore, the solution can be obtained as

$$u(x,t)=-\frac{\sqrt{3(2k^2-1)}}{2k}\tanh k(x+t+d/k). \quad (20)$$

$$u(x,t)=-\frac{\sqrt{3(2k^2-1)}}{2k}\coth k(x+t+d/k). \quad (21)$$

Case-IV: When we choose $A=-1, B=0, C=1$ and $k\neq\lambda$ then we can found coefficients. Therefore, the solution is

$$u(x,t)=\pm\frac{5i\sqrt{(\lambda-k)}}{k^{3/2}\sqrt{1992}}\mp\frac{83i}{6\sqrt{k(\lambda-k)}\sqrt{1992}}\tanh^2(kx+\lambda t+d), \quad (22)$$

$$u(x,t)=\pm\frac{5i\sqrt{(\lambda-k)}}{k^{3/2}\sqrt{1992}}\mp\frac{83i}{6\sqrt{k(\lambda-k)}\sqrt{1992}}\coth^2(kx+\lambda t+d). \quad (23)$$

Case-V: When we choose $A=1$ and $B=0$ then the Riccati equation (3) has the following solutions

$$\psi=\begin{cases}-\sqrt{-C}\tan\left(\sqrt{-C}\eta\right), & C<0,\\ 1/\eta, & C=0,\\ \sqrt{C}\tan\left(\sqrt{C}\eta\right), & C>0.\end{cases} \tag{24}$$

In this way, we can successfully recover the previously known traveling wave solutions that had been obtained by the HBM method and other method.

We obtained the following traveling wave solution from equations. (4), (13a)-(13c) and (24)

$$u(x,t)=\pm\frac{5i\sqrt{k^2C^2+\lambda^2+2C^2\lambda k}}{C^{3/2}k^{1/2}\sqrt{1992(k+\lambda)}}$$
$$-5\left[\left(\frac{\mp 31ik^{3/2}\sqrt{C^2k^2+\lambda^2+2C^2\lambda k}}{\sqrt{23904C^2(k+\lambda)^2-24025k^3(k^2C^2+\lambda^2+2C^2\lambda k)}}\right)\mp\frac{12(k+\lambda)^{3/2}C^{1/2}\sqrt{1992k}}{\sqrt{23904C^2(k+\lambda)^2-24025k^3(k^2C^2+\lambda^2+2C^2\lambda k)}}\right]C\tanh^2(\sqrt{-C}(kx+\lambda t+d)) \tag{25}$$

$$u(x,t)=\pm\frac{5i\sqrt{k^2C^2+\lambda^2+2C^2\lambda k}}{C^{3/2}k^{1/2}\sqrt{1992(k+\lambda)}}$$
$$+5\left[\left(\frac{\mp 31ik^{3/2}\sqrt{C^2k^2+\lambda^2+2C^2\lambda k}}{\sqrt{23904C^2(k+\lambda)^2-24025k^3(k^2C^2+\lambda^2+2C^2\lambda k)}}\right)\mp\frac{12(k+\lambda)^{3/2}C^{1/2}\sqrt{1992k}}{\sqrt{23904C^2(k+\lambda)^2-24025k^3(k^2C^2+\lambda^2+2C^2\lambda k)}}\right]C\tanh^2(\sqrt{C}(kx+\lambda t+d)) \tag{26}$$

$$u(x,t)=\pm\frac{5i\sqrt{k^2C^2+\lambda^2+2C^2\lambda k}}{C^{3/2}k^{1/2}\sqrt{1992(k+\lambda)}}$$
$$+5\left[\left(\frac{\mp 31ik^{3/2}\sqrt{C^2k^2+\lambda^2+2C^2\lambda k}}{\sqrt{23904C^2(k+\lambda)^2-24025k^3(k^2C^2+\lambda^2+2C^2\lambda k)}}\right)\mp\frac{12(k+\lambda)^{3/2}C^{1/2}\sqrt{1992k}}{\sqrt{23904C^2(k+\lambda)^2-24025k^3(k^2C^2+\lambda^2+2C^2\lambda k)}}\right]\frac{1}{(kx+\lambda t+d)^2} \tag{27}$$

3 CONCLUSIONS

In this summary, we have applied homogeneous balance method to obtain traveling wave solutions of the Fornberg-Whitham equation. At the same time, these new exact solutions will enrich previous results and help us further understand the physical structures and analyze the nonlinear propagation of the traveling waves. Method used in this paper is quite efficient to solve differential equations, especially for the HBM method, it is rather heuristic and gives multiple-soliton solutions directly for a wide class of nonlinear equations. The HB method is very valid to study the solutions for the integrable system and can also be applied to other nonlinear evolution equations in mathematical physics. This is our task in future work.

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