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Recent Development In Fixed-Point Theory, Optimization, And Their Applications

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Abstract
Variational and linear inequalities, approximation theory, nonlinear analysis, integral and differential equations and inclusions, dynamic systems theory, mathematics of fractals, mathematical economics (game theory, equilibrium problems, and optimisation problems), and mathematical modelling all rely on fixed point theory. The fixed point of some mapping F is the solution to many problems in pure and applied mathematics. As a result, a variety of procedures in numerical analysis and approximation theory result in successive approximations to the fixed point of an approximate mapping.

Keyword
Fixed point theory, game theory,applications,quality management ,

Introduction

In many different domains, fixed point theory has played a significant role since LuitzenBrouwer's discovery of the first known fixed point theorem in 1909. It's possible to find several examples in the fields of optimization and approximation theory; differential equations; variational inequalities; and supplementary problems. In mathematics and other fields such as biology, engineering, physics, computer sciences, data sciences, economics, etc., Fixed Point Theory is an integrative theory that provides discernment and significant instruments for the solution of specific issues. For single-valued or set-valued mappings of metric spaces, topological vector spaces, posets and lattices, and Banach lattices, fixed point theorems are derived. For showing the presence of fixed points in mappings, it's crucial to understand approximation of fixed points of certain maps. Solvability of optimization and differential equations can be proven using it. Variational inequalities and equilibrium issues can also be shown to be solveable using it.

There is a lot of interest in nonlinear analysis and optimization due to the importance and volume of research being done, as well as the availability of numerous new methods for studying them that use fixed point approximations.

It is important to know about fixed point theory not only because it is used in the theory of partial differential equations, integral equations, differential inclusions, and random differential equations (e.g. Rus, Petruşel, Petruşel, 2008; Longa, Nieto and Son, 2016), but also because it is used in approximation methods (e.g. Petruşel and Yao, 2009; Mishra, Pant and Panicker, 2016). (i.e. Isac,Yuan, Tan, Yu, 1998; Rus, Iancu, 2000; Song, Guo, Chen, 2016). In 1973, H. Scarf proposed the first constructive approach for computing the fixed point of a continuous function. Boyd and Wong (1969), Hardy and Rogers(1973), Husain and Sehgal(1975), Caristi (1976), and Downing and Kirk(1977) all made generalisations of Banach's fixed point theorem in different directions. The source paper by Rhoades provides a comprehensive comparison of several definitions and fixed point theorems for contraction and contractive maps (1977).

Fixed point

When a point undergoes a specified transformation yet does not change, it is said to be a fixed point.

Let T : K → K be a mapping. The point $x \in K$ is called a fixed point of T if x is mapped onto itself i.e. $T_x = x$.
Example.

1. A translation does not have a starting or ending point.
2. There is just one fixed point in a planar rotation. The only thing that will never move is the centre of rotation.

There may be no fixed point in a mapping, a single fixed point, several fixed points, or an infinite number of fixed points in a mapping.

i. The mapping $x \to x^2$ of R into itself has two fixed point. Indeed, the point 0 and 1 are fixed point.
ii. The projection $(x, y) \to x$ of R^2 onto the x-axis has infinitely many fixed points. If fact, all points of x-axis are fixed points.

Fixed Point Theorems

Maps f of a set X into itself that, under particular conditions, admit a fixed point, i.e., a point $x \in X$ such that $f(x) = x$ are known as fixed point theorems. Knowing whether or not there are fixed points is useful in a wide range of areas of analysis and topology. Let's look at an example that's both easy and instructive.
Example
Suppose we are given a system of n equations in n unknowns of the form gj $(x_1, \ldots, x_n) = 0$, $j = 1, \ldots, n$
where the g_j are continuous real-valued functions of the real variables x_j.
Leth_j $(x_1, \ldots, x_n) = g_j$ $(x_1, \ldots, x_n) + x_j$, and for any point $x = (x_1, \ldots, x_n)$ define
$h(x) = (h_1(x), \ldots, h_n(x))$. Assume now that h has a fixed point $\bar{x} \in R^n$.
Then it is easily seen that $\bar{x}$ is a solution to the system of equations.

.Fixed Point Theory Application

Well-known examples of how fixed points can be used in best approximation theory, mini-max problem solving, mathematical economics, and variations in inequalities. [Zeidler (1986)].

Location of zeros

Let X, Y be Banach spaces, and $f : B_X(x_0, r) \to Y$ be a Fr´echet differentiable map. In order to find a zero for f, the idea is to apply aniterative method constructing a sequence x_n (starting from x0) so that x_{n+1} is the zero of the tangent of f at x_n. Assuming that $f^0(x)^{-1} \in L(Y, X)$ on $B_X(x_0, r)$, one has x

$$x_{n+1} = x_n - f'(x_n)^{-1} f(x_n)$$

Theorem

Let X, Y be Banach spaces, and $f : B_X(x_0, r) \to Y$ be a Fr´echet differentiable map. Assume that, for some $\lambda > 0$,

(a) $f^0(x_0)$ is invertible;
(b) $||f'(x) - f'(x_0)||L(X,Y) \leq \lambda||x - x_0||_X, \quad \forall\, x \in B_X(x_0, r)$;
(c) $\mu := 4\lambda||f'(x_n)^{-1}||^2_{L(X,Y)}||f(x_0)||_Y \leq 1$;
(d) $s := 2||f'(x_n)^{-1}||\ L(X,Y)||f(x_0)||_Y < r$.

Then there exists a unique $\bar{x} \in B_X(x_0, s)$ such that $f(\bar{x}) = 0$.

$$\big||\Phi'(x)|\big|L(X) \leq \big||f'(x_0)^{-1}|\big|L(Y,X)\big||f'(x_0) - f'(x)|\big|L(X,Y)$$
$$\leq \lambda s\big||f'(x_0)^{-1}|\big|L(Y,X) = \frac{\mu}{2}$$

Hence Φ is Lipschitz, with Lipschitz constant less than or equal to $\mu/2 \leq 1/2$. Moreover,

$$||\Phi(\mathrm{x}) - x_0||x \leq ||\mathrm{f}'(x_0)^{-1}||L(Y,X)||f(x_0)||Y = \frac{s}{2}$$

which in turn gives

$$||\Phi(\mathrm{x}) - x_0||x \leq ||\Phi(x) - \Phi(x_0)||X + ||\Phi(x_0) - x_0||X \leq \frac{\mu}{2}||x - x_0||X + \frac{s}{2} \leq s.$$

Hence Φ is a contraction on BX(x0, s). FromBenach Theorem there exists a unique x¯ ∈ B_X(x0, s) such that Φ(¯x) = ¯x, which implies f(¯x) = 0.

Concerning the convergence speed of x_n to ¯x, by virtue of the remark after Benach Theorem, we get

$$||x_n - \bar{x}||X \leq \frac{s\mu^n}{(2-\mu)2^n}.$$

Also, since

$$x_{n+1} - \bar{x} = \mathrm{f}'(x_0)^{-1}(\mathrm{f}'(x_0) - \mathrm{f}'(x_n))(x_n - \bar{x}) + o(||x_n - \bar{z}||x)$$

it follows that

$$||x_{n+1} - \bar{x}||x = \frac{\mu}{2}||x_n - \bar{x}||x + o(||x_n - \bar{z}||x).$$

Hence

$$||x_{n+1} - \bar{x}||x \leq c||x_n - \bar{x}||x$$

for some c ∈ (0, 1). for all large n. This is usually referred to as linear convergence of the method.

Game Theory.

The challenges linked to the analysis of the quality of a tangible or intangible product may be handled, in some situations, as problems from the game theory. Cooperative and noncooperative video games are also possible. At least one of the Nash equilibrium points in a finite noncooperative game has a finite number of components, as demonstrated by Kohlberg and Mertens in 1986. Nonlinear problems can benefit from using important components, according to Yu and Yang in 2004. For a Ky-Fan inequality and a contraction mapping in Hilbert space, Vuong, Strodiot, and Nguyen (2012) present some new iterative methods for discovering a common member of the set of points meeting it.

We analyse a game in which there are n ≥ 2 participants and no cooperation is assumed between them. Each player has a strategy, which is influenced by the other players' tactics. K_k represents the set of all feasible tactics for the kth player, and K = $K_1 \times \cdots \times K_n$ represents the set of all players' strategies. A strategy profile is a set of x ∈ K elements. Let f_k: K → R be the k^{th} player's loss function for each k.If

$$\sum_{k=1}^{n} f_k(x) = 0, \qquad \forall x \in K$$

A zero-sum game is one in which no points are gained or lost. Each player's goal is to minimise his or her loss, or to maximise his or her gain, depending on your perspective.

Research Methodology

Theorem 1.1 (Brouwer's Fixed Point Theorem)

Whenever the closed unit ball S = {x : ||x||≤ 1} is continuously being mapped from one location in Rn to another, a fixed point will be found. To put it another way, "any continuous mapping of a closed convex set in Rn into itself has a fixed point." is an analogous statement. His theorem was first put forth in 1912 by Brouwer, a Dutch mathematician. Proofs of this fundamental theorem can be found, but they all use Algebraic Topology as their starting point. However, Sasty and Bram, Bers, Kantorovich and Akilvo can be used as proof. It should be noted that theorems like this one, where the spaces are Rn subsets, aren't very useful in functional analysis, where the focus is on infinite dimensional subsets of function spaces. Birkhoff and Kellog were the first to look into this in 1922 while working on the Existence Theorem in analysis. Using a compact convex subset of C[0, 1] and $L^2[0, 1]$, they found fixed points for continuous self-mappings. A Polish mathematician named Schauder went on to make a formal statement on his theorem in 1930. Schauder extended these findings to compact convex subsets of normed linear spaces.

Theorem 1.2 (Schauder's Fixed Point Theorem)

In normed space X, let C be a compact non-empty convex subset of C. Then there is a fixed point in every continuous mapping from C to C. Approximation is used to prove Schauder's theorem by first approximating the infinite-dimensional set C to the finite-dimensional set, then using Brouwer's theorem to show that the finite-dimensional approximation has a fixed point, and finally by taking the limit of this approximation as the dimension increases into the infinitesimal range.

Theorem 1.3 (Schauder-Tychonoff)

Let K be a nonvoid compact convex subset of a finite dimensional real Banach space X. Then every continuous function $f : K \to K$ has a fixed point.

In the real Banach space X, let K be a compact nonvoid convex subset. Then there is a fixed point for every continuous function $f : K \to K$.

$\bar{x} \in K$.

Since X is homeomorphic to $R^n$for some $n \in N$, we assume without loss of generality $X = R^n$. Also, we can assume $K \subset D^n$. For every $x \in D^n$,

let $p(x) \in K$ be the unique point of minimum norm of the set $x - K$. Notice that $p(x) = x$ for every $x \in K$. Moreover, p is continuous on D^n. Indeed, givenx_n, $x \in D^n$, with $x_n \to x$,

$$||x - p(x)|| \leq ||x - p(x_n)|| \leq ||x - x_n|| + \inf_{k \in K} ||x_n - k|| \to ||x - p(x)||$$

as $n \to \infty$. Thus $x - p(x_n)$ is a minimizing sequence as $x_n \to x$ in $x - K$, and this implies the convergence $p(x_n) \to p(x)$. Define now $g(x) = f(p(x))$. Then g maps continuously D^n onto K.

As an immediate application, consider Example. If there is a compact and convex set $K \subset R^n$such that $h(K) \subset K$, then h has a fixed point $\bar{x} \in K$.

The Banach contraction principle

Definition

Let X be a metric space equipped with a distance d. A map $f : X \to X$ is said to be Lipschitz continuous if there is $\lambda \geq 0$ such that $d(f(x_1), f(x_2)) \leq \lambda d(x_1, x_2)$, $\forall\ x_1, x_2 \in X$.The smallest λ for which the above inequality holds is the Lipschitz constant of f. If $\lambda \leq 1$ f is said to be non-expansive, if $\lambda < 1$ f is said to be a contraction.

Theorem [Banach]

Let f be a contraction on a complete metric space X.

Then f has a unique fixed point $\bar{x} \in X$.

Notice first that if $x_1, x_2 \in X$ are fixed points of f, then

$d(x_1, x_2) = d(f(x_1), f(x_2)) \leq \lambda d(x_1, x_2)$

which imply $x_1 = x_2$. Choose now any $x_0 \in X$, and define the iterate sequence

$x_{n+1} = f(x_n)$. By induction on n,

$d(x_{n+1}, x_n) \leq \lambda^n d(f(x_0), x_0)$.

If $n \in N$ and $m \geq 1$,

$$d(x_{n+m}, x_n) \leq d(x_{n+m}, x_{n+m-1}) + \cdots + d(x_{n+1}, x_n)$$
$$\leq (\lambda^{n+m} + \cdots + \lambda^n) d(f(x_0), x_0)$$
$$\leq \frac{\lambda^n}{1 - \lambda} d(f(x_0), x_0).$$

Hence x_n is a Cauchy sequence, and admits a limit $\bar{x} \in X$, for X is complete.

Since f is continuous, we have $f(\bar{x}) = \lim_n f(x_n) = \lim_n x_{n+1} = \bar{x}$.

Sequences of maps and fixed points

We'll assume that the metric space is (X, d). Our focus is on the converging fixed points issue for a succession of maps f_n, where each map is X.

Theorem

Assume that each f_n has at least a fixed point $x_n = f_n(x_n)$. Let $f : X \to X$ be a uniformly continuous map such that f m is a contraction for somem ≥ 1. If f_n converges uniformly to f, then x_n converges to $\bar{x} = f(\bar{x})$.proof We first assume that f is a contraction (i.e., m = 1). Let $\lambda < 1$ be the Lipschitz constant of f. Given $\varepsilon > 0$, choose $n_0 = n_0(\varepsilon)$ such that

$$d\big(f_n(x), f(x)\big) \le \varepsilon(1-\lambda), \qquad \forall\, n \ge n_0, \forall\, x \in X .$$

Then, for $n \ge n_0$,

$$d(x_n, \bar{x}) = d(f_n(x_n), f(\bar{x}))$$
$$\le d\big(f_n(x_n), f(x_n)\big) + d(f(x_n), f(\bar{x}))$$
$$\le \varepsilon(1-\lambda) + \lambda d(x_n, \bar{x}).$$

Therefore $d(x_n, \bar{x}) \le \varepsilon$, which proves the convergence.

To prove the general case it is enough to observe that if

$d(f^m(x), f^m(y)) \le \lambda^m d(x, y)$for some $\lambda < 1$, we can define a new metric d_0 on X equivalent to d by setting

$$d_0(x,y) = \sum_{k=0}^{m-1} \frac{1}{\lambda^k} d\left(f^k(x), f^k(y)\right).$$

Moreover, since f is uniformly continuous, f_n converges uniformly to f also with respect to d_0. Finally, f is a contraction with respect to d_0. Indeed,

$$d_0(f(x), f(y)) = \sum_{k=0}^{m-1} \frac{1}{\lambda^k} d\left(f^{k+1}(x), f^{k+1}(y)\right)$$

$$= \lambda \sum_{k=1}^{m-1} \frac{1}{\lambda^k} d\left(f^k(x), f^k(y)\right) + \frac{1}{\lambda^{m+1}} d\big(f^m(x), f^m(y)\big)$$

$$\le \lambda \sum_{k=0}^{m-1} \frac{1}{\lambda^k} d\left(f^k(x), f^k(y)\right) = \lambda d_0(x,y).$$

So the problem is reduced to the previous case m = 1.

Fixed points of non-expansive maps

Let X be a Banach space, $C \subset X$ nonvoid, closed, bounded and convex, and let $f : C \to C$ be a non-expansive map. The problem is whether f admits a fixed point in C. The answer, in general, is false.

Let $X = c_0$ with the supremum norm. Setting $C = B_X(0, 1)$, themap $f : C \to C$ defined by

$f(x) = (1, x_0, x_1, \ldots)$, for $x = (x_0, x_1, x_2, \ldots) \in C$

is non-expansive but clearly admits no fixed points in C.

Things are quite different in uniformly convex Banach spaces.

Theorem [Browder-Kirk]

Let X be a uniformly convex Banach space and $C \subset X$ be nonvoid, closed, bounded and convex. If $f : C \to C$ is a non-expansivemap, then f has a fixed point in C.

Let $x_* \in C$ be fixed, and consider a sequence $r_n \in (0, 1)$ converging to 1. For each $n \in \mathbb{N}$, define the map $f_n : C \to C$ as

$$f_n(x) = r_n f(x) + (1 - r_n)x_* .$$

Notice that f_n is a contractions on C, hence there is a unique $x_n \in C$ such that $f_n(x_n) = x_n$. Since C is weakly compact, x_n has a subsequence (still denoted byx_n) weakly convergent to some $\bar{x} \in C$. We shall prove that $\bar{x}$ is a fixed point of f. Notice first that

$$\lim_{n\to\infty}(||f(\bar{x})-x_n||^2-\big||\bar{x}-x_n||^2\big)=||f(\bar{x})-\bar{x}||^2 .$$

Since f is non-expansive we have

$$\big||f(\bar{x})-x_n\big|| \le \big||f(\bar{x})-f(x_n)\big|| + ||f(x_n)-x_n||$$
$$\le \big||\bar{x}-x_n\big|| + ||f(x_n)-x_n||$$
$$= \big||\bar{x}-x_n\big|| + (1-r_n)||f(x_n)-x_*||$$

But $r_n \to 1$ as $n \to \infty$ and C is bounded, so we conclude that

$$\lim\sup_{n\to\infty}(||f(\bar{x})-x_n||^2-||\bar{x}-x_n||^2)\le 0$$

which yields the equality f(¯x) = x¯

Proposition In the hypotheses of Theorem , the set F of fixed points of f is closed and convex.

Proof

The first assertion is trivial. Assume then $x_0, x_1 \in F$, with $x_0 \neq x_1$, and denote $x_t = (1 - t)x_0 + tx_1$, with $t \in (0, 1)$. We have

$$\big||f(x_t)-x_0\big|| = \big||f(x_t)-f(x_0)\big|| \le \big||x_t-x_0\big|| = t||x_1-x_0||$$
$$\big||f(x_t)-x_1\big|| = \big||f(x_t)-f(x_1)\big|| \le \big||x_t-x_1\big|| = (1-t)||x_1-x_0||$$

that imply the equalities

$$\big||f(x_t)-x_0\big|| = t||x_1-x_0||$$
$$\big||f(x_t)-x_1\big|| = (1-t)\big||x_1-x_0\big||$$

The proof is completed if we show that $f(x_t) = (1-t)x_0 + t_{x1}$. This follows from a general fact about uniform convexity, which is recalled in the next lemma

Lemma

Let X be a uniformly convex Banach space, and let $\alpha, x, y \in X$ be such that

$$\big||\alpha-x\big|| = t\big||x-y\big||, \qquad \big||\alpha-y\big|| = (1-t)\big||x-y\big||,$$

For some $t \in [0,1]$. Then $\alpha = (1-t)x + ty$.

Proof

Without loss of generality, we can assume $t \ge 1/2$. We have

$$\big||(1-t)(\alpha-x)-t(\alpha-y)\big|| = \big||(1-2t)(\alpha-x)-t(x-y)\big||$$
$$\ge t\big||x-y\big|| - (1-2t)||\alpha-x||$$
$$= 2t(1-t)\big||x-y\big|| .$$

Since the reverse inequality holds as well, and

$$(1-t)\big||\alpha-x\big|| = t\big||\alpha-y\big|| = t(1-t)||x-y||$$

from the uniform convexity of X (but strict convexity would suffice) we get

$$\big||\alpha-(1-t)x-ty\big|| = \big||(1-t)(\alpha-x)+t(\alpha-y)\big|| = 0$$

as claimed

Review of Literature

Several key single-valued mappings have fixed points, and their results can be applied in engineering, physics, computer science, economics, and communications (Alfuraidan& Ansari, 2016).

Fixed point theorems for non-expansive mappings were established by Browder(1965), Gohde, and Kirk(1966).. The proof of fixed point theorems can be divided into two categories. One approach proves the existence of fixed points, whereas the other approximates the point as the limit of an iterative series. They defined acceptable set and proved numerous fixed point theorems using the concept of a convex hull in metric spaces, which they generalised by Krik and colleagues (1972). To replace the positive real number set with an ordered Banach space, Huang and Zhang (2007) recently presented the concept of a cone metric space. It was shown by Rezapour and Hamilbarani(2008) that a non-normal cone metric space exists, and that some fixed point theorems can be found in these spaces.

It was Rao et al. who first focused on complex valued b-metric spaces, which were broader than complex valued metric spaces, to get fixed point findings. Many authors have followed up on this work by demonstrating some fixed point findings for various mappings that satisfy rational requirements with respect to complex valued b-metric spaces and the associated references. There have recently been results achieved by substituting the constant of contractive condition for the control functions in complex valued metric spaces by Sintunavaratet al.,Sitthikul and Saejung, and Singhetal. Many authors have proven various normal fixed point results for a few mappings that satisfy more general contractive criteria, including rational expressions with point-subordinate control functions as coefficients in complex valued b-metric spaces.

Conclusion

In the last few decades, the fixed point theory has found numerous uses. Optimisation theory, game theory, conflict situations and mathematical quality modelling all benefit greatly from its applications in these fields.

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