SOME COMMON FIXED POINT THEOREMS FOR TWO PAIRS OF WEAKLY COMPATIBLE SELF-MAPS IN METRIC SPACES

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ABSTRACT: *In the present manuscript, we shall prove common fixed point theorems for two pairs of weakly compatible self mappings with E.A. property and (CLR) property. Our results extends and unifies some results present in the literature.*

Keywords: metric space, weakly compatible mapping, E.A. property, (CLR) property.

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1. INTRODUCTION

Fixed point theory is one of the most fruitful and applicable topics of nonlinear analysis, which is widely used not only in other mathematical theories, but also in many practical problems of natural Sciences and Engineering. The Banach contraction mapping principle is indeed the most popular result of metric fixed point theory. This principle has many applications in several domains, such as differential equations, functional equations, integral equations, economics, wild life, and several others.

Branciari gave an integral version of the Banach contraction principles and proved fixed point theorem for a single-valued contractive mapping of integral type in metric space. Afterwards many researchers extended the result of Baranciari and obtained fixed point and common fixed point theorems for various contractive conditions of integral type on different spaces. Now, we recollect some known definitions and results from the literature which are helpful in the proof of our main results.

Definition1.1: A coincidence point of a pair of self-mapping $A, B : X \to X$ is a point $x \in X$ for which $Ax = Bx$.

A common fixed point of a pair of self-mapping $A, B: X \to X$ is a point $x \in X$ for which $Ax = Bx = x$. Jungck initiated the concept of weakly compatible maps to study common fixed point theorems.

Definition1.2: A pair of self-mapping $A, B: X \to X$ is weakly compatible if they commute at their coincidence points, that is, if there exists a point $x \in X$ such that $ABx = BAx$ whenever $Ax = Bx$.

In the study of common fixed points of weakly compatible mappings, we often require the assumption of completeness of the space or subspace or continuity of mappings involved besides some contractive condition. Aamri and Moutawakil [1] introduced the notion of E.A. property, which, requires only the closedness of the subspace and Liu *et al.* extended the E.A. property to common the E.A. property as follows:

Definition 1.3: Let(X, d) be a metric space and $A, B, P, Q: X \to X$ be four self-maps. The pairs (A, Q) and (B, P) satisfy the common E.A. property if there exist two sequence $\{x_n\}$ and $\{y_n\}$ in X such that $\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Qx_n = \lim_{n\to\infty} By_n = \lim_{n\to\infty} Py_n = s \in X$.

Sintunavarat and Kumam introduced the notion of the (CLR) property, which never requires any condition on closedness of the space or subspace and Imdad *et al.* introduced the common(CLR) property ehich is an extension of the (CLR) property.

Definition1.4: Let (X, d) be a metric space and $A, B, P, Q: X \to X$ be four self-maps. The pairs (A, Q) and (B, P) satisfy the common limit range property with respect to mappings Q and Pdenoted by (CLR_{PQ}) if there exists two sequences $\{x_n\}$ and $\{y_n\}$ in X such that

$\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Qx_n = \lim_{n\to\infty} By_n = \lim_{n\to\infty} Py_n =$ s$\in QX \cap PX$.

Lemma1.5: Let (X, d) be a metric space and $\{x_n\}$ be a sequence in X. Then $\{x_n\}$ converges to x if and only if $|d(x_n, x)| \to 0$ as $n \to \infty$, where $m \in \mathbb{N}$.

Jungck [4] introduced the concept of weakly compatible maps as follows:

Definition1.6: Let f and g be two self-mappings of a metric space(X, d). Then a pair (f, g) is said to be weakly compatible if they commute at coincidence points.

In 2002, Aamari and Moutawakil [1] introduced the notion of E.A. property as follows.

Definition1.7: Let (X, d) be a complete metric space and $T: X \to X$ be a contraction mapping, i.e. $d(Tx, Ty) \leq \alpha d(x, y)\, \forall\, x, y \in X$ and $\alpha \in$[0,1) then T has a unique fixed point.

Definition 1.8: Let f and g be two self-mappings of a metric space (X, d). Then a pair (f, g) is said to satisfy E.A. property if there exists a sequence $\{x_n\}$ in X such that $\lim_{n\to\infty} fx_n$ = $\lim_{n\to\infty} gx_n$ = fx for some $x \in X$.

Example 1.9: Suppose X = [2,4] with $\delta(\alpha, \beta) = e\hat{}(|\alpha - \beta|)\, \forall\, \alpha, \beta \in X$.

Define $G(\alpha) = \{\, 2\; if\; \alpha = 2, 2\alpha/3\; if\; 3 < \alpha \leq 4\}$and

$I(\alpha) = \{\, 2\; if\; 2 \leq \alpha \leq 3, \alpha + 3/3$ if $3 \leq \alpha < 4\}$.

Take a sequence $\{\alpha_k\}$ as α_k = $3 + 1/k\; for\; k \geq 0$. Then $G\alpha_k = G$(3+1/k) = 2(3+1/k)/3 = 2+1/k = 2 as k→ ∞ and $I\alpha_k = I(3 + 1/k) = (3 + 1/k + 3)/3 = (6/3 + 1/3k) = 2 + 1/k = 2$ as k→ ∞. This gives $G\alpha_k = I\alpha_k = 2 \in X$ as $k \to \infty$. This gives (G, I) satisfies E.A. property.

Definition 1.10: Let X be a non-empty set. Then f and g over a metric space(X, d) satisfy CLR property if $\lim_{n\to\infty} f(x_n)$ = $\lim_{n\to\infty} g(x_n)$ = $g(t)$ for some $t \in X$.

Example 1.11: Let $X = \mathbb{R}$. Define the mapping $d: X \times X \to \mathbb{R}$ by $d(z_1, z_2) = 2|z_1, z_2|$ for all z_1,$z_2 \in X$. Then $(X, d)$is a metric space. Define S and $T: X \to X$ by $Sz = z$ and $Tz = 2z$ for all $z \in X$, respectively. Consider a sequence $\{z_n\}$ = $\{\frac{1}{n}\}(n \in \mathbb{N})$ in X. Then

$\lim_{n\to\infty} Sz_n$ = $\lim_{n\to\infty} z_n$= 0 and $\lim_{n\to\infty} Tz_n$ = $\lim_{n\to\infty} 2z_n$ = 0.

Thus, S and T satisfy CLR_s property.

The following definitions will be used in sequel:

Let Φ_1 be the set of all functions φ : [0, ∞) → [0, ∞) satisfying the following conditions:

1. ϕ is continuous on [0, ∞).
2. ϕ is non-decreasing.
3. ϕ(0) = 0 and ϕ(t) > 0 for all t > 0.

Let Φ_3 be the set of all functions α : [0, ∞) → [0, 1) satisfying the following conditions:

1. α is measurable or continuous on [0, ∞).
2. α(0) = 0 and α(t) < 1 for every t > 0.
3. Optionally, sup_{t>0} α(t) = κ < 1 (a uniform contraction bound) or α is non-decreasing.

2. Fixed Point Theorems for Weakly Compatible Mappings with E.A. Property

Now, we prove common fixed point theorems for two pairs of weakly compatible self-maps along with E.A. property.

Theorem 2.1: Let A, B, S and T be self mapping in a metric space (X, d) such that

(C1) $SX \subset BX$ and $TX \subset AX$

(C2) (A, S) and (B, T) are weakly compatible;

(C3) $\int_0^{d(Sx,Ty)} \varphi(t)dt \leq \alpha(d(x, y)) \int_0^{M_1(x,y)} \varphi(t)dt, \forall x, y \in X$

Where $(\phi , \alpha) \in \Phi_1 \times \Phi_3$ and for all $x, y \in X$.

$M_1(x, y) = \max\{d(Ax, By), d(Ax, Sx), d(By, Ty), \frac{1}{2}[d(Ax, Ty) + d(Sx, By)], td(Ax, Sx), \frac{1+d(Ax,By)}{1+d(Ax,Sx)} d(By, Ty), \frac{d^2(Ax,Sx)}{1+d(Sx,Ty)}, \frac{d^2(By,Ty)}{1+d(Sx,Ty)}, \frac{1+d(Ax,Ty)+d(Sx,By)}{1+d(Ax,By)+d(Sx,Ty)} d(Ax, Sx), \frac{1+d(Ax,Ty)+d(Sx,By)}{1+d(Ax,By)+d(Sx,Ty)} d(By, Ty).$

(C4) The pairs (A, S) and (B, T) satisfy the E.A. property.

Suppose that any one of AX, BX, SX, TX is a closed subspace of X. Then A, B, S and T have a unique common fixed point.

Proof. Suppose that (A, S) satisfies the E.A. property. Then there exists a sequence $\{x_n\}$ in X such that Ax_n = Sx_n= z for some $z \in X$.

Since $SX \subset BX$, there exists a sequence $\{y_n\}$ in X such that Sx_n = By_n. Hence, $\lim_{n\to\infty} By_n$ = z.

We shall show that $\lim_{n\to\infty} By_n$ = z. We shall show that $\lim_{n\to\infty} Ty_n$ = z. Let $\lim_{n\to\infty} Ty_n$ = t ≠z.

Put $x = x_n, y = y_n$

$\int_0^{d(Sx_n,Ty_n)} \varphi(t)dt \leq \alpha(d(x,y) \int_0^{M_1(x,y)} \varphi(t)dt, \forall x,y \in X.$

$M_1(x_n, y_n)$ = max{d(Ax_n, By_n),d(Ax_n,Sx_n),d(By_n,Ty_n),$\frac{1}{2}$[d(Ax_n,Ty_n)+d(Sx_n,By_n)], $\frac{1+d(Ax_n,By_n)}{1+d(By_n,Ty_n)} d(Ax_n, Sx_n)$, $\frac{1+d(Ax_n,By_n)}{1+d(Ax_n,Sx_n)} d(By_n, Ty_n)$, $\frac{d^2(Ax_n,Sx_n)}{1+d(Sx_n,Ty_n)}$, $\frac{d^2(By_n,Ty_n)}{1+d(Sx_n,Ty_n)}$, $\frac{1+d(Ax_n,Ty_n)+d(Sx_n,By_n)}{1+d(Ax_n,By_n)+d(Sx_n,Ty_n)} d(Ax_n, Sx_n)$, $\frac{1+d(Ax_n,Ty_n)+d(Sx_n,By_n)}{1+d(Ax_n,By_n)+d(Sx_n,Ty_n)} d(By_n, Ty_n)$}

Now, $\lim_{n\to\infty} Ax_n$ = $\lim_{n\to\infty} Sx_n$ = z = $\lim_{n\to\infty} By_n$ and $\lim_{n\to\infty} Ty_n = t$.

$M_1(x_n, y_n)$ = max{$d(z,z)$, $d(z,z), d(z,t)$, $\frac{1}{2}[d(z,t) + d(z,z)]$,$\frac{1+d(z,z)}{1+d(z,t)} d(z,z)$, $\frac{1+d(z,z)}{1+d(z,z)} d(z,t)$, $\frac{d^2(z,z)}{1+d(z,t)}$, $\frac{d^2(z,t)}{1+d(z,t)}$, $\frac{1+d(z,t)+d(z,z)}{1+d(z,t)+d(z,t)} d(z,z)$, $\frac{1+d(z,t)+d(z,z)}{1+d(z,z)+d(z,t)}$d(z,t)}

= max {0, 0, $d(z,t)$,0, $d(z,t)$,0, $d^2(z,t)$, $d(z,t)$}

= $d(z,t)$,

where,

$\lim_{n\to\infty} \int_0^{d(Sx_n,Ty_n)} \varphi(t)dt \leq \alpha(d(x_n,y_n) \int_0^{M_1(x_n,y_n)} \varphi(t)dt$

$\int_0^{d(Sz,Tz)} \varphi(t)dt \leq \alpha(d(z,t) \int_0^{M_1(x_n,y_n)} \varphi(t)dt$

$\leq \alpha(d(z,t) \int_0^{d(z,t)} \varphi(t)dt,$

which is contradiction.

Therefore, t = z. i.e. $\lim_{n\to\infty} Ty_n$ = z.

Suppose that BX is a closed space of X. Then there exists $u \in X$ such that $z = Bu$.

Subsequently, we have

$\lim_{n\to\infty} Ty_n = \lim_{n\to\infty} Sx_n = \lim_{n\to\infty} Ax_n = \lim_{n\to\infty} By_n = z = Bu.$

Now, we shall show that $Tu = Bu$.

Let $Tu \neq Bu$.

From (C3), we have

$\int_0^{d(Sx_n,Ty_n)} \varphi(t)dt \le \alpha(d(x_n,y_n)) \int_0^{M_1(x_n,y_n)} \varphi(t)dt, \forall x,y \in X$

Letting n→ ∞

$\lim_{n\to\infty} \int_0^{d(Sx_n,z)} \varphi(t)dt \le \alpha(d(x_n,y_n)) \int_0^{M_1(x_n,y_n)} \varphi(t)dt, \forall x,y \in X.$

$$\int_0^{d(Sx_n,Bu)} \varphi(t)dt \le \alpha(d(x_n,y_n))\int_0^{M_1(x_n,y_n)} \varphi(t)dt, \tag{2.1}$$

where,

$M_1(x_n,y_n) = \max\{d(Ax_n,By_n),\ d(Ax_n,Sx_n), d(By_n,Ty_n), \frac{1}{2}[d(Ax_n,Ty_n)+d(Sx_n,By_n)],$
$\frac{1+d(Ax_n,By_n)}{1+d(By_n,Ty_n)} d(Ax_n,Sx_n),\ \frac{1+d(Ax_n,By_n)}{1+d(Ax_n,Sx_n)} d(By_n,Ty_n),\ \frac{d^2(Ax_n,Sx_n)}{1+d(Sx_n,Ty_n)},\ \frac{d^2(By_n,Ty_n)}{1+d(Sx_n,Ty_n)},$
$\frac{1+d(Ax_n,Ty_n)+d(Sx_n,By_n)}{1+d(Ax_n,By_n)+d(Sx_n,Ty_n)} d(Ax_n,Sx_n), \frac{1+d(Ax_n,Ty_n)+d(Sx_n,By_n)}{1+d(Ax_n,By_n)+d(Sx_n,Ty_n)} d(By_n,Ty_n)\}$

$= \max\{d(Ax_n,Bu),\ d(Ax_n,Sx_n),\ d(Bu,Tu),\ \frac{1}{2}[d(Ax_n,Ty_n)+d(Sx_n,By_n)],$
$\frac{1+d(Ax_n,By_n)}{1+d(By_n,Ty_n)} d(Ax_n,Sx_n),\ \frac{1+d(Ax_n,By_n)}{1+d(Ax_n,Sx_n)} d(By_n,Ty_n),\ \frac{d^2(Ax_n,Sx_n)}{1+d(Sx_n,Tu)},\ \frac{d^2(Bu,Tu)}{1+d(Sx_n,Tu)},$
$\frac{1+d(Ax_n,Tu)+d(Sx_n,Bu)}{1+d(Ax_n,Bu)+d(Sx_n,Tu)} d(Ax_n,Sx_n), \frac{1+d(Ax_n,Tu)+d(Sx_n,Bu)}{1+d(Ax_n,Bu)+d(Sx_n,Tu)} d(By_n,Tu)$

$= \max\{0,0,d(Bu,Tu),0,0,d(Bu,Tu),0,d^2(Bu,Tu),0,d(Bu,Tu)\}$

$$= d(z,Tu) \tag{2.2}$$

Equation (2.1) and (2.2) leads to contradiction. Therefore, $Tu = z = Bu$.

Since B and T are weakly Compatible, we have $Btu = TBu$. Hence, $TTu = TBu = Btu = BBu$.

Since $TX \subset AX$, there exists, $v \in X$ such that $Tu = Av$.

Now, we claims that $Av = Sv$. Let $Av \neq Sv$.

$M_1(v,u) = \max\{d(Av,Bu), d(Av,Sv), d(Bu,Tu), \frac{1}{2}[d(Av,Tu) + d(Su,Bu)], \frac{1+d(Av,Bu)}{1+d(Bu,Tu)} d(Av,Sv), \frac{1+d(Av,Bu)}{1+d(Av,Sv)} d(Bu,Tu), \frac{d^2(Av,Sv)}{1+d(Sv,Tu)}, \frac{d^2(Bu,Tu)}{1+d(Sv,Tu)}, \frac{1+d(Av,Tu)+d(Sv,Bu)}{1+d(Av,Bu)+d(Sv,Tu)} d(Av,Sv), \frac{1+d(Av,Tu)+d(Sv,Bu)}{1+d(Av,Bu)+d(Sv,Tu)} d(Bu,Tu)\}$

= max{0,$d(Av,Sv)$,0,0,$d(Av,Sv)$,0,$d^2(Av,Sv)$,0,$d(Av,Sv)$,0}

= $d(Av,Sv) = d(Tu,Sv)$

Thus from (C3), we have

$\int_0^{d(Sv,Tu)} \varphi(t)dt \leq \alpha(\text{d(v,u)}) \int_0^{M_1(v,u)} \varphi(t)dt, \forall x,y \in X.$

$\leq \alpha(\text{v,u}) \int_0^{d(Sv,Tu)} \varphi(t)dt.$

Which is a contradiction. Therefore, $Sv = Tu = Av$.

Thus, we have $Tu = Bu = Sv = Av$. The weak compatibility of A and S implies that $ASv = SAv = SSv = AAv$. Now, we claim that Tu is the common fixed point of A, B, S and T. Suppose that $TTu \neq Tu$. From (C3), we have $d(Tu,TTu) = d(Sv,TTu)$

$\int_0^{d(Sv,TTu)} \varphi(t)dt \leq \alpha(\text{d(v,Tu)}) \int_0^{M_1(v,Tu)} \varphi(t)dt, \forall x,y \in X.$

and

$M_1(v,Tu) = \max\{d(Av,Btu), d(Av,Sv), d(BTu,TTu), \frac{1}{2}[d(Av,TTu) + d(Sv,BTu)], \frac{1+d(Av,BTu)}{1+d(BTU,TTu)} d(Av,Sv), \frac{1+d(Av,BTu)}{1+d(Av,Sv)} d(BTu,TTu), \frac{d^2(Av,Sv)}{1+d(Sv,TTu)}, \frac{d^2(BTu,TTu)}{1+d(Sv,TTu)}, \frac{1+d(Av,TTu)+d(Sv,BTu)}{1+d(Av,BTu)+d(Sv,TTu)} d(Av,Sv), \frac{1+d(Av,TTu)+d(Sv,BTu)}{1+d(Av,BTu)+d(Sv,TTu)} d(BTU,TTu)\}$

= max{$d(Av,BTu), d(Av,Sv), d(BTU,TTu)$,0,0,0,0,0,0,0}

= $d(Av,BTu) = d(Tu,TTu)$.

Thus, from equation (3) leads to a contradiction. Therefore, $Tu = TTu = BTu$.

Hence Tu is the common fixed point of B and T. Similarly, we prove that Sv is the common fixed point of A and S. Since $Tu = Sv$, Tu is the common fixed point of A, B, S and T.

The proof is similar when AX is assumed to be a closed subspace of X. The cases in which TX or SX is a complete subspace of X are similar to the cases in which $AX\ or\ BX$, respectively is complete subspaces of X. Since $TX \subset AX$ and $SX \subset BX$.

Finally, for uniqueness, let p and q $(p \neq q)$ be two common fixed points of A, B, S and T.

From (C3), we have

$$\int_0^{d(Sp,Tq)} \varphi(t)dt \leq \alpha(d(\text{p,q}))\int_0^{M_1(p,q)} \varphi(t)dt, \forall x, y \in X. \tag{2.4}$$

Where,

$$M_1(p,q) = \max\{d(Ap,Bq), d(Ap,Sp), d(Bq,Tq), \tfrac{1}{2}[d(Ap,Tq) + d(Sp,Bq)], \frac{1+d(Ap,Bq)}{1+d(Bq,Tq)}d(Ap,Sp), \frac{1+d(Ap,Bq)}{1+d(Ap,Sp)}d(Bq,Tq), \frac{d^2(Ap,Sp)}{1+d(Sp,Tq)}, \frac{d^2(Bq,Tq)}{1+d(Sp,Tq)}, \frac{1+d(Ap,Tq)+d(Sp,Bq)}{1+d(Ap,Bq)+d(Sp,Tq)}d(Ap,Sp), \frac{1+d(Ap,Tq)+d(Sp,Bq)}{1+d(Ap,Bq)+d(Sp,Tq)}d(Bq,Tq)\}$$

= max$\{d(Ap,Bq), d(Ap,Sp), d(Bq,Tq)$,0,0,0,0,0,0,0$\}$

= $d(p,q)$.

From equation (2.4) which leads to contradiction. Therefore, $p = q$. Hence A, B, S and T have a unique common fixed point. This completes the proof.

If $A = B$ and $S = T$ in the above Theorem,we get the following:

Corollary 2.2: Let A and S be two self-mappings of a complex valued metric space (X, d) satisfying $SX \subset Ax$,

$$\int_0^{d(Sx,Ty)} \varphi(t)dt \leq \alpha(\text{d(x,y)})\int_0^{M_1(x,y)} \varphi(t)dt, \forall x, y \in X$$

M_1(x,y) = max$\{d(Ax, By), d(Ax, Sx), d(By, Ty), \frac{1}{2}[(d(Ax, Ty) + d(Sx, By)],$
$\frac{1+d(Ax,By)}{1+d(By,Ty)} d(Ax, Sx),$ $\frac{1+d(Ax,By)}{1+d(Ax,Sx)} d(By, Ty),$ $\frac{d^2(Ax,Sx)}{1+d(Sx,Ty)},$ $\frac{d^2(By,Ty)}{1+d(Sx,Ty)},$
$\frac{1+d(Ax,Ty)+d(Sx,By)}{1+d(Ax,By)+d(Sx,Ty)} d(Ax, Sx), \frac{1+d(Ax,Ty)+d(Sx,By)}{1+d(Ax,By)+d(Sx,Ty)} d(By, Ty)\}$

The pair (A, S) is weakly compatible. If one of AX or SX is closed subspaces of X, then A and S have a unique common fixed point.

3. Fixed Point Theorems for Weakly Compatible Mappings with CLR Property

Now, we prove common fixed point theorems for weakly compatible mappings with CLR property.

Theorem 3.1: Let A, B, S and T be four self-mappings of a complex valued metric space (X, d) satisfying (C2), (C3), and (C9) $SX \subset BX$ and the pairs (A, S) satisfies CLR_A property or $TX \subset AX$ and the pair (B, T) satisfies CLR_B property. Then A, B, S and T have a unique common fixed point.

Proof: Without loss of generality, assume that $SX \subset BX$ and the pairs (A, S) satisfies CLR_A property. Then there exists a sequence $\{x_n\}$ in X s.t. $\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Sx_n$= Ax for some $x \in X$. Since $SX \subset BX$, there exists a sequence $\{y_n\}$ in X such that $Sx_n = By_n$.

Hence $\lim_{n\to\infty} By_n = Ax$.

We shall show that $\lim_{n\to\infty} Ty_n = Ax$. Let $\lim_{n\to\infty} Ty_n$= z $\neq Ax$.

Form (C3) we have

$$\int_0^{d(Sx_n,Ty_n)} \varphi(t)dt \leq \alpha(\mathrm{d}(x_n, y_n)) \int_0^{M_1(x_n,y_n)} \varphi(t)dt, \forall x, y \in X.$$

Letting n→ ∞, we have

$$\int_0^{d(Ax,z)} \varphi(t)dt \leq \alpha(\mathrm{d}(x_n, y_n)\int_0^{M_1(x_n,y_n)} \varphi(t)dt, \quad (3.1)$$

where

$M_1(x_n, y_n)=$ max{d(Ax_n, By_n), d(A$x_{n,}Sx_n$), d(By_n,Ty_n), $\frac{1}{2}[d(Ax_nTy_n)$+d(Sx_n,By_n)], $\frac{1+d(Ax_n,By_n)}{1+d(By_n,Ty_n)}d(Ax_n,Sx_n)$, $\frac{1+d(Ax_n,By_n)}{1+d(Ax_n,Sx_n)}d(By_n,Ty_n)$, $\frac{d^2(Ax_n,Sx_n)}{1+d(Sx_n,Ty_n)}$, $\frac{d^2(By_n,Ty_n)}{1+d(Sx_n,Ty_n)}$, $\frac{1+d(Ax_n,Ty_n)+d(Sx_n,By_n)}{1+d(Ax_n,By_n)+d(Sx_n,Ty_n)}d(Ax_n, Sx_n)$, $\frac{1+d(Ax_n,Ty_n)+d(Sx_n,By_n)}{1+d(Ax_n,By_n)+d(Sx_n,Ty_n)}d(By_n, Ty_n)$

Letting limit n tends to infinity.

$\lim_{n\to\infty} M_1(x_n, y_n)=$ $\lim_{n\to\infty} \max\{d(Ax, Ax), d(Ax, Ax), d(Ax, z), \frac{1}{2}[d(Ax, z) +$ *dAx,Ax],* *1+dAx,Ax1+dAx,zd(Ax,Ax),* *1+dAx,Ax1+dAx,Axd(Ax,z),* $\frac{d^2(Ax,Ax)}{1+d(Ax,Z)}, \frac{d^2(Ax,Z)}{1+d(Ax,Z)}, \frac{1+d(Ax,Z)+d(Ax,Ax)}{1+d(Ax,Ax)+d(Ax,Z)}d(Ax, Ax), \frac{1+d(Ax,Z)+d(Ax,Ax)}{1+d(Ax,Ax)+d(Ax,Z)}d(Ax, Z)\}$

$= \lim_{n\to\infty} \max\{0,0, d(Ax, z), \frac{1}{2}d(Ax, z), 0, d(Ax, z), 0, \frac{d^2(Ax,z)}{1+d(Ax,z)}, 0, d(Ax, z)\} = d(Ax, z).$

Thus, from equation(3.1) this leads to contradiction. Therefore, Ax=z and hence $\lim_{n\to\infty} Ty_n = Ax$.

Subsequently, we have

$\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Sx_n = \lim_{n\to\infty} By_n = \lim_{n\to\infty} Ty_n = Ax = z.$

Now, we shall show that $Sx = z$. Let $Sx \neq z$.

From (C3), we have

$\int_0^{d(Sx_n,Ty_n)} \varphi(t)dt \leq \alpha(d(x,y_n))\int_0^{M_1(x,y_n)} \varphi(t)dt, \forall x, y \in X,$

where,

$M_1(x, y_n)$ = max{d(Ax,By_n), d(Ax,Sx), d(By_n, Ty_n), $\frac{1}{2}[d(Ax, Ty_n) + d(Sx, By_n)]$, $\frac{1+d(Ax,By_n)}{1+d(By_n,Ty_n)}d(Ax, Sx)$, $\frac{1+d(Ax,By_n)}{1+d(Ax,Sx_n)}d(By_n,Ty_n)$, $\frac{d^2(Ax,Sx)}{1+d(Sx,Ty_n)}$, $\frac{d^2(By_n,Ty_n)}{1+d(Sx,Ty_n)}$, $\frac{1+d(Ax,Ty_n)+d(Sx,By_n)}{1+d(Ax,By_n)+d(Sx,Ty_n)}d(Ax, Sx)$, $\frac{1+d(Ax,Ty_n)+d(Sx,By_n)}{1+d(Ax,By_n)+d(Sx,Ty_n)}d(By_n,Ty_n)\}$

= max{d(z,z), d(z,Sx), d(z,z), $\frac{1}{2}[d(z,z)+d(Sx,z)]$, $\frac{1+d(z,z)}{1+d(z,z)}d(z,Sx)$, $\frac{1+d(z,z)}{1+d(z,z)}d(z,z)$, $\frac{d^2(z,Sx)}{1+d(z,z)}$, $\frac{d^2(z,z)}{1+d(Sx,z)}$, $\frac{1+d(z,z)+d(Sx,z)}{1+d(z,z)+d(Sx,z)}d(z,z)$,$\frac{1+d(z,z)+d(Sx,z)}{1+d(z,z)+d(Sx,z)}d(z,z)$}

= max{0,$d(z,Sx)$,0,$\frac{1}{2}d(Sx,z)$, $d(z,Sx)$, 0,$d^2(z,Sx)$,0,0,0}

= $d(Ax,z)$

Thus,

$\int_0^{d(Sx,Ty_n)}\varphi(t)dt \le \alpha$(d(x,$y_n$)) $\int_0^{M_1(x,y_n)}\varphi(t)dt,\ \forall x,y\in X$

$\int_0^{d(Sx,z)}\varphi(t)dt \le \alpha$(d(x,$y_n$)) $\int_0^{d(Ax,z)}\varphi(t)dt,$

a contradiction.

Therefore, $Sx = z = Ax$.

Since the pair (A,S) is weakly compatible, it follows that $Az = Sz$. Also, since $SX \subset BX$, there exists $y\in X$ such that $z = Sx = By$. Now, we show that $Ty = z$.

Let $Ty \neq z$. From (C3), we have

$\int_0^{d(Sx_n,Ty_n)}\varphi(t)dt \le \alpha$(d($x_n,y_n$)) $\int_0^{M_1(x_n,y_n)}\varphi(t)dt$, $\forall x,y\in X$.

Letting n→ ∞

$\int_0^{d(Ax,z)}\varphi(t)dt \le \alpha$(d($x_n,y_n$)) $\int_0^{M_1(x_n,y_n)}\varphi(t)dt$, ∀x,y∈X. (3.2)

$M_1(x_n,y)$ = max {$d(Ax_n,By)$, d(Ax_n,Sx_n), d(By,Ty), $\frac{1}{2}$[d(Ax_n,Ty)+d(Sx_n,By)], $\frac{1+d(Ax_n,By)}{1+d(By,Ty)}d(Ax_n,Sx_n)$, $\frac{1+d(Ax_n,By)}{1+d(Ax_n,Sx_n)}d(By,Ty)$, $\frac{d^2(Ax_n,Sx_n)}{1+d(Sx_n,Ty)}$, $\frac{d^2(By,Ty)}{1+d(Sx_n,Ty)}$, $\frac{1+d(Ax_n,Ty)+d(Sx_n,By)}{1+d(Ax_n,By)+d(Sx_n,Ty)}d(Ax_n,Sx_n)$, $\frac{1+d(Ax_n,Ty)+d(Sx_n,By)}{1+d(Ax_n,By)+d(Sx_n,Ty)}d(By,Ty)$}

= max{d(z,z),d(z,z),d(z,Ty), $\frac{1}{2}[d(z,Ty)+d(z,z)]$, $\frac{1+d(z,z)}{1+d(z,Ty)}d(z,z)$, $\frac{1+d(z,z)}{1+d(z,z))}d(z,Ty)$, $\frac{d^2(z,z)}{1+d(z,Ty)}$, $\frac{d^2(z,Ty)}{1+d(z,z)}$, $\frac{1+d(z,Ty)+d(z,z)}{1+d(z,z)+d(z,Ty)}d(z,z)$, $\frac{1+d(z,Ty)+d(z,z)}{1+d(z,z)+d(z,Ty)}d(z,Ty)$} = d(z,Ty)

Thus from equation (3.2)

$\int_0^{d(Ax,z)} \varphi(t)dt \le \alpha(\mathrm{d}(x_n, y) \int_0^{d(z,Ty)} \varphi(t)dt$

Which is a contradiction. Thus, $z = Ty = By$.

Since the pair (B, T) is weakly compatible, it follows that $Tz = Bz$. Now, we claim that $Sz = Tz$.

Let $Sz \neq Tz$. From (C3) we have

$\int_0^{d(Sz,Tz)} \varphi(t)dt \le \alpha(\mathrm{d(z,z)}\int_0^{M_1(z,z)} \varphi(t)dt,$ (3.3)

where,

$M_1(z,z) = \max\{\mathrm{d(Az,Bz)}, \mathrm{d(Az,Sz)}, \mathrm{d(Bz,Tz)}, \frac{1}{2}[\mathrm{d(Az,Tz)+d(Sz,Bz)}], \frac{1+d(Az,Bz)}{1+d(Bz,Tz)}d(Az,Sz), \frac{1+d(Az,Bz)}{1+d(Az,Sz)}d(Bz,Tz), \frac{d^2(Az,Sz)}{1+d(Sz,Tz)}, \frac{d^2(Bz,Tz)}{1+d(Sz,Tz)}, \frac{1+d(Az,Tz)+d(Sz,Bz)}{1+d(Az,Bz)+d(Sz,Tz)}d(Az,Sz), \frac{1+d(Az,Tz)+d(Sz,Bz)}{1+d(Az,Bz)+d(Sz,Tz)}d(Bz,Tz)\}$

$= \max\{d(Sz,Tz), d(Sz,Sz), d(Tz,Tz), \frac{1}{2}[d(Sz,Tz)+d(Sz,Tz)], \frac{1+d(Sz,Tz)}{1+d(Tz,Tz)}d(Sz,Sz), \frac{1+d(Sz,Tz)}{1+d(Sz,Sz)}d(Tz,Tz), \frac{d^2(Sz,Sz)}{1+d(Sz,Tz)}, \frac{d^2(Tz,Tz)}{1+d(Sz,Tz)}, \frac{1+d(Sz,Tz)+d(Sz,Tz)}{1+d(Sz,Tz)+d(Sz,Tz)}d(Sz,Sz), \frac{1+d(Sz,Tz)+d(Sz,Tz)}{1+d(Sz,Tz)+d(Sz,Tz)}d(Sz,Tz)\} = d(Sz,Tz)$

Thus , from (3.3), we have

$\int_0^{d(Sz,Tz)} \varphi(t)dt \le \alpha(\mathrm{d(z,z))}\int_0^{M_1(z,z)} \varphi(t)dt = 0$

Which is a contradiction. Therefore, $Sz = Tz$, i.e. $Az = Sz = Tz = Bz$.

Now we shall show that z = Tz. Let z ≠Tz. From (C3), we have

$\int_0^{d(z,Tz)} \varphi(t)dt = \int_0^{d(Sx,Tz)} \varphi(t)dt \le \alpha(\mathrm{x,z})\int_0^{M_1(x,z)} \varphi(t)dt,$ (3.4)

where,

$M_1(x,z)$ = max{d(Ax,Bz), d(Ax,Sz), d(Bx,Tz) , $\frac{1}{2}[d(Ax,Tz)+d(Sx,Bz)]$, $\frac{1+d(Ax,Bz)}{1+d(Bz,Tz)}d(Ax,Sx)$, $\frac{1+d(Ax,Bz)}{1+d(Ax,Sx)}d(Bz,Tz)$, $\frac{d^2(Ax,Sx)}{1+d(Sx,Tz)}$, $\frac{d^2(Bz,Tz)}{1+d(Sx,Tz)}$, $\frac{1+d(Ax,Tz)+d(Sx,Bz)}{1+d(Ax,Bz)+d(Sx,Tz)}d(Ax,Sx)$, $\frac{1+d(Ax,Tz)+d(Sx,Bz)}{1+d(Ax,Bz)+d(Sx,Tz)}d(Bz,Tz)$}

= max{d(z,Tz) , d(z,Tz),d(z,Tz),$\frac{1}{2}[d(z,Tz)+d(z,Bz)]$, $\frac{1+d(z,Bz)}{1+d(Bz,Tz)}d(z,z)$, $\frac{1+d(z,Bz)}{1+d(z,z)}d(Bz,Tz)$, $\frac{d^2(z,z)}{1+d(z,Tz)}$, $\frac{d^2(Bz,Tz)}{1+d(z,Tz)}$, $\frac{1+d(z,Tz)+d(z,Bz)}{1+d(z,Bz)+d(z,Tz)}d(z,z)$, $\frac{1+d(z,Tz)+d(z,Bz)}{1+d(z,Bz)+d(z,Tz)}d(Bz,Tz)$}

= max{$d(z,Tz), d(z,Tz), d(z,Tz), d(z,Tz)$,0,0,0,0,0,0}

= $d(z,Tz)$

Thus , from (3.4)

$\int_0^{d(z,Tz)}\varphi(t)dt = \int_0^{d(Sx,Tz)}\varphi(t)dt \le \alpha(x,z)\int_0^{d(z,Tz)}\varphi(t)dt$

Which is a contradiction.

Therefore, $z = Tz = Bz = Az = Sz$. Hence z is the common fixed point of A, B, S and T.

Finally, for uniqueness, let u(u$\neq z$) be another common fixed point of A, B, S and T.

$\int_0^{d(Su,Tz)}\varphi(t)dt = \int_0^{d(u,z)}\varphi(t)dt \le \alpha(d(u,z))\int_0^{M_1(u,z)}\varphi(t)dt\ \forall x,y \in X$.

where,

$M_1(u,z)$ = max{d(Au,Bz), d(Au,Su), d(Bz,Tz), $\frac{1}{2}[d(Au,Tz)+d(Su,Bz)]$, $\frac{1+d(Au,Bz)}{1+d(Bz,Tz)}d(Au,Su)$, $\frac{1+d(Au,Bz)}{1+d(Au,Su)}d(Bz,Tz)$, $\frac{d^2(Au,Su)}{1+d(Su,Tz)}$, $\frac{d^2(Bz,Tz)}{1+d(Su,Tz)}$, $\frac{1+d(Au,Tz)+d(Su,Bz)}{1+d(Au,Bz)+d(Su,Tz)}d(Au,Su)$, $\frac{1+d(Au,Tz)+d(Su,Bz)}{1+d(Au,Bz)+d(Su,Tz)}d(Bz,Tz)$}

= $d(u,z)$.

Thus, from (3.4)

$\int_0^{d(Su,Tu)} \varphi(t)dt \leq \int_0^{d(u,z)} \varphi(t)dt \leq \alpha(d(u,z)) \int_0^{d(u,z)} \varphi(t)dt \; \forall x, y \in X.$

Which is a contradiction.

Therefore, $u = z$. Hence A, B, S and T have a unique common fixed point. This Completes the proof.

From Theorem 3.1 , if $A = B$ and $S = T$, we get the following.

Corollary 3.2: Let A and S be two self –mappings of a complex valued metric space (X, d) satisfying

(C6) $SX \subset AX$

(C7) $\int_0^{d(Sx,Sy)} \varphi(t)dt \leq \alpha(\mathrm{d(x,y)})\int_0^{M_1(x,y)} \varphi(t)dt$, $\forall x, y \in X$.

where,

$M_1(x,y) = \max\{\mathrm{d(Ax,Ay),d(Ax,Sx),d(Ay,Sy)}, \frac{1}{2}[d(Ax,Sy) + d(Sx,Ay)], \frac{1+d(Ax,Ay)}{1+d(Ay,Sy)} d(Ax,Sx),$ $\frac{1+d(Ax,Ay)}{1+d(Ax,Sx)} d(Ay,Sy),$ $\frac{d^2(Ax,Sx)}{1+d(Sx,Sy)},$ $\frac{d^2(Ay,Sy)}{1+d(Sx,Sy)},$ $\frac{1+d(Ax,sy)+d(Sx,Ay)}{1+d(Ax,Ay)+d(Sx,Sy)} d(Ax,Sx),$ $\frac{1+d(Ax,Sy)+d(Sx,Ay)}{1+d(Ax,Ay)+d(Sx,Sy)} d(Ay,Sy)\},$

for each x, y in X.

(C8) the pair (A, S) is wekly compatible.

(C9) the pair (A, S) satisfies CLR_A property. Then A and S have a unique common fixed point.

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