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RATIONAL CONTRACTIVE CONDITIONS AND ANALYSIS OF COMMON FIXED POINTS USING WEAK COMPATIBLE MAPPINGS IN MULTIPLICATIVE METRIC SPACES

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ABSTRACT. *This paper is part of a series of papers proving the common fixed point result for four mappings using rational contractive conditions, in which the pairs of maps are assumed to satisfy weak commutativity in the setup of multiplicative metric spaces.*

KEYWORDS. Complete multiplicative metric space, multiplicative contraction, weak compatible mappings and commuting mappings

AMS Subject Classification. 47H10, 54H25

1. INTRODUCTION

Bashirov et al. [1] introduced the concept of Multiplicative calculus and its applications in 2008, where they discussed why the non-Newtonian calculi of Grossman and Katz are equally important as the calculus of Newton and Leibniz. non-Newtonian calculi of Grossman and Katzare alternatives to the classical calculus of Newton and Leibniz. Grossman and Katz provide a wide variety of mathematical tools that are used worldwide in science, engineering, and mathematics. It is noteworthy that there are infinitely many non-Newtonian calculi, Ozavsar and Cevikel [2] introduced the concept of multiplicative metric space in 1991, replacing the usual triangular inequality with Multiplicative triangle inequality " $d(x, y) \leq d(x, z).d(z, y)$ for all $x, y, z \in X$". It is well established that mathematical results are very useful for evaluating the presence and uniqueness of different mathematical models with regard to some types of contraction mappings. Inspired by this work, Isha et al. [3] proved Common Fixed Point Theorems Governed by Rational Inequalities and Intimate Mappings in Multiplicative Metric Spaces. As a generalisation of commuting maps (fg=gf), the concept of compatible mappings was introduced in [4].

The study of common fixed point theorem from the various class of class of mappings was initiated by various authors [5, 6, 7, 8]. Considering the importance of non-Newtonian calculus, Yong [9] et al. proved common fixed point theorems for weakly compatible mapping satisfying implicit functions in multiplicative metric spaces, whereas Sharma et al. [10] proved common fixed point theorems usingrational contractive condition in multiplicative metric spaces. Readers are requested to refer [11-19, 22] to understand the existing results of fixed point theorems. He et al. [20] also studied the common fixed point theorems for weak commutative mapping on a multiplicative metric space, which itself are the note worthy results of non-Newtonian calculus. The utility of compatibility in the context of fixed point theory was demonstrated by extending a Park and Bae theorem [23]. The primary objective of this note is to further emulate the compatible map concept. We extend the following strong result of S.L. Singh and S.P. Singh [24] by using compatible maps instead of commuting maps and four functions instead of three. There are a number of implementations of having studied fixed points mapping that meet specific contraction conditions and numerous research activities have been focused on. We introduced the concept of multiplicative mapping with distinct approach of contractions and proved some common fixed-point theorems in the framework of multiplicative spaces. We generalized some distinct fixed point theorems in the context of multiplicative metric spaces in this article. We used the concept of multiplicative contraction mapping to prove some fixed point theorems on complete multiplicative metric spaces. It is important to note that the concept of commuting maps has proven useful for generalizing in the context of metric space fixed point theory (see, e.g., [23-34]).To prove our results and make this article self-contained, few existing definitions and results are important to mention.

Definition 1.1 [2] Let X be a non-empty set. Multiplicative metric is a mapping $d : X \times X \to \mathbb{R}^+$ satisfying the following conditions:

(M1) $d(x,y) \geq 1$ for all x, y ∈ X and d(x, y) = 1 iff x = y,

(M2) d(x, y) = d(y, x) for all x, y ∈ X,

(M3) d(x, z) ≤ d(x, y)·d(y, z) for all x, y, z ∈ X (multiplicative triangle inequality).

Then d is called a Multiplicative metric on X and (X, d) is called a **multiplicative metric space**.

Proposition 1.2 [1] Let (X,d) be a multiplicative metric space, $\{x_n\}$be a sequence in X and x ∈ X. Then $x_n \to x\ (n \to \infty)$ if and only if $d(x_n, x) \to 1\ (n \to \infty)$.

Definition 1.3 [1] Let (X,d) be a multiplicative metric space and$\{x_n\}$be a sequence in X .Then$\{x_n\}$ is a called multiplicative Cauchy sequence if and only if $d(x_n, x_m) \to 1\ (n,m \to \infty)$

Definition 1.4[4] The self-maps f and g of a multiplicative metric space (X,d) are said to be compatible if $\lim_{n\to\infty} d(fgx_n gfx_n) = 1$, whenever$\{x_n\}$ is a sequence in X such that $\lim_{n\to\infty} fx_n = \lim_{n\to\infty} gx_n = t$, for some $t \in X$.

Definition 1.5 [2] Suppose that f and g are two self-maps of a multiplicative metric space (X,d). The pair (f,g) are called weakly compatible mappings if $fx = gx, x \in X$ implies $fgx = gfx$. That is, $d(fx, gx) = 1 \Rightarrow d(fgx, gfx) = 1$.

2. Main Result:

The concept of implicit functions is used by Popa [21] which is an effective contractive condition in multiplicative metric space. Here, we define a suitable class of the implicit function involving five real non-negative arguments as follows:

Let Ψ denote the family of functions such that $\phi: (\mathbb{R}^+)^5 \to \mathbb{R}^+$ is continuous and increasing in each coordinate variable and

i. $\phi(t, t.t_1, 1, t.t_1, t) \leq t.t_1$
ii. $\phi(t, 1, t.t_1, t.t_1, t_1) \leq t.t_1$
iii. $\phi(1, t, 1, t, 1) \leq t$
iv. $\phi(t, 1, t, t, 1) \leq t$
v. $\phi(t, t, t, t, 1) \leq t$

for every $t, t_1 \in \mathbb{R}^+$ $(t, t_1 \geq 1)$. It is obvious that $\phi(1, 1, 1, 1, 1) = 1$. There exist many functions $\phi \in \Psi$.

The following theorem is the generalized result of C. Yong Jung et.al [2] for pairs of weakly compatible mappings using rational contraction map satisfying implicit functions in multiplicative metric space.

Theorem 2.1 Let A, B, S, T be mappings of a multiplicative metric space (X, d) into itself satisfying the following conditions:

(2.1) $SX \subset BX$, $TX \subset AX$,

$$(2.2) \quad d(Sx, Ty)$$

$$\leq \left\{ \phi \left\{ \begin{array}{c} \frac{d(Ax, By)[d(Ax, Sx) + d(Ty, Sx)]}{d(By, Ty) + d(By, Ax)}, \frac{d(Ax, By)[d(Ty, Sx) + d(Ax, Ty)]}{d(By, Ax) + d(Sx, By)}, \\ \frac{d(Ty, Sx)[d(By, Ax) + d(Sx, By)]}{d(Ty, Sx) + d(Ax, Ty)}, \frac{d(Ty, By)d(Ty, Sx)[d(Ax, Sx) + d(Ax, By)]}{d(Sx, Ty) + d(Ty, By)}, \\ \frac{d(Ax, Sx)[d(By, Ty) + d(By, Ax)]}{d(Ax, Sx) + d(Ty, Sx)} \end{array} \right\} \right\}^{\lambda}$$

for all $x, y \in X$, where $\lambda \in (0, \frac{1}{2})$ and $\phi \in \Psi$;

(2.3) let us suppose that the pairs (A, S) and (B, T) are weakly compatible and

(2.4) one of the subspaces AX or BX or SX or TX is complete.

Then A, B, S and T have a unique common fixed point.

Proof. Let x_0 be any arbitrary point of metric space X. It is given that $SX \subset BX$, hence there exists $x_1 \in X$ such that $Sx_0 = Bx_1 = y_0$. Now for this x_1 there exists $x_2 \in X$ in such a way that $Ax_2 = Tx_1 = y_1$. In a similar way, we can define an inductive sequence $\{ y_n \}$ in such a way that,

(2.5) $Sx_{2n} = Bx_{2n+1} = y_{2n}$, $Ax_{2n+2} = Tx_{2n+1} = y_{2n+1}$

Next, we prove that $\{ y_n \}$ is a multiplicative cauchy sequence in X. In fact, $\forall\, n \in \mathbb{N}$,

using equations (2.2) and (2.5), we have

$$d(y_{2n}, y_{2n+1}) \leq d(Sx_{2n}, Tx_{2n+1})$$

$$\leq \left\{ \phi \left\{ \begin{array}{c} \dfrac{d(Ax_{2n}, Bx_{2n+1})[d(Ax_{2n}, Sx_{2n}) + d(Tx_{2n+1}, Sx_{2n})]}{d(Bx_{2n+1}, Tx_{2n+1}) + d(Bx_{2n+1}, Ax_{2n})}, \\ \dfrac{d(Ax_{2n}, Bx_{2n+1})[d(Tx_{2n+1}, Sx_{2n}) + d(Ax_{2n}, Tx_{2n+1})]}{d(Bx_{2n+1}, Ax_{2n}) + d(Sx_{2n}, Bx_{2n+1})}, \\ \dfrac{d(Tx_{2n+1}, Sx_{2n})[d(Bx_{2n+1}, Ax_{2n}) + d(Sx_{2n}, Bx_{2n+1})]}{d(Tx_{2n+1}, Sx_{2n}) + d(Ax_{2n}, Tx_{2n+1})}, \\ \dfrac{d(Tx_{2n+1}, Bx_{2n+1})d(Tx_{2n+1}, Sx_{2n})[d(Ax_{2n}, Sx_{2n}) + d(Ax_{2n}, Bx_{2n+1})]}{d(Sx_{2n}, Tx_{2n+1}) + d(Tx_{2n+1}, Bx_{2n+1})}, \\ \dfrac{d(Ax_{2n}, Sx_{2n})[d(Bx_{2n+1}, Tx_{2n+1}) + d(Bx_{2n+1}, Ax_{2n})]}{d(Ax_{2n}, Sx_{2n}) + d(Tx_{2n+1}, Sx_{2n})} \end{array} \right\} \right\}^{\lambda}$$

$$\leq \left\{ \phi \left\{ \begin{array}{c} \dfrac{d(y_{2n-1}, y_{2n})[d(y_{2n-1}, y_{2n}) + d(y_{2n+1}, y_{2n})]}{d(y_{2n}, y_{2n+1}) + d(y_{2n}, y_{2n-1})}, \\ \dfrac{d(y_{2n-1}, y_{2n})[d(y_{2n+1}, y_{2n}) + d(y_{2n-1}, y_{2n+1})]}{d(y_{2n}, y_{2n-1}) + d(y_{2n}, y_{2n})}, \\ \dfrac{d(y_{2n+1}, y_{2n})[d(y_{2n}, y_{2n-1}) + d(y_{2n}, y_{2n})]}{d(y_{2n+1}, y_{2n}) + d(y_{2n-1}, y_{2n+1})}, \\ \dfrac{d(y_{2n+1}, y_{2n})d(y_{2n+1}, y_{2n})[d(y_{2n-1}, y_{2n}) + d(y_{2n-1}, y_{2n}))]}{d(y_{2n}, y_{2n+1}) + d(y_{2n+1}, y_{2n})}, \\ \dfrac{d(y_{2n-1}, y_{2n})[d(y_{2n}, y_{2n+1}) + d(y_{2n}, y_{2n-1})]}{d(y_{2n-1}, y_{2n}) + d(y_{2n+1}, y_{2n})} \end{array} \right\} \right\}^{\lambda}$$

$$\leq \left\{\phi\left\{\begin{array}{c}\dfrac{d(y_{2n-1},y_{2n})[d(y_{2n-1},y_{2n})+d(y_{2n+1},y_{2n})]}{d(y_{2n},y_{2n+1})+d(y_{2n},y_{2n-1})},\\ \dfrac{d(y_{2n-1},y_{2n})[d(y_{2n+1},y_{2n})+d(y_{2n-1},y_{2n}).d(y_{2n},y_{2n+1})]}{d(y_{2n},y_{2n-1})+1},\\ \dfrac{d(y_{2n+1},y_{2n})[d(y_{2n},y_{2n-1})+1]}{d(y_{2n+1},y_{2n})+d(y_{2n-1},y_{2n}).d(y_{2n},y_{2n+1})},\\ \dfrac{d(y_{2n+1},y_{2n})d(y_{2n+1},y_{2n})[d(y_{2n-1},y_{2n})+d(y_{2n-1},y_{2n}))]}{d(y_{2n},y_{2n+1})+d(y_{2n+1},y_{2n})}\\ \\ \dfrac{d(y_{2n-1},y_{2n})[d(y_{2n},y_{2n+1})+d(y_{2n},y_{2n-1})]}{d(y_{2n-1},y_{2n})+d(y_{2n+1},y_{2n})}\end{array}\right\}\right\}^{\lambda}$$

$$\leq \left\{\phi\left\{\begin{array}{c}d(y_{2n-1},y_{2n}),\\ d(y_{2n-1},y_{2n}).d(y_{2n},y_{2n+1}),\\ 1,\\ d(y_{2n-1},y_{2n}).d(y_{2n},y_{2n+1}),\\ d(y_{2n-1},y_{2n})\end{array}\right\}\right\}^{\lambda}$$

$d(y_{2n},y_{2n+1}) \leq d^{\lambda}(y_{2n-1},y_{2n}).d^{\lambda}(y_{2n},y_{2n+1})$ [using (i)]

This implies that,

$$d(y_{2n},y_{2n+1}) \leq d^{\frac{\lambda}{1-\lambda}}(y_{2n-1},y_{2n})$$

On substituting, $h = \frac{\lambda}{1-\lambda} < 1$, since $\lambda \in (0,\frac{1}{2})$

$$(2.6) \quad d(y_{2n},y_{2n+1}) \leq d^{h}(y_{2n-1},y_{2n})$$

In a similar way we have,

$d(y_{2n+1},y_{2n+2}) = d(Tx_{2n+1},Sx_{2n+2}) = d(Sx_{2n+2},Tx_{2n+1})$

$$\leq \left\{\phi\left\{\begin{array}{c}\dfrac{d(Ax_{2n+2},Bx_{2n+1})[d(Ax_{2n+2},Sx_{2n+2})+d(Tx_{2n+1},Sx_{2n+2})]}{d(Bx_{2n+1},Tx_{2n+1})+d(Bx_{2n+1},Ax_{2n+2})},\\ \dfrac{d(Ax_{2n+2},Bx_{2n+1})[d(Tx_{2n+1},Sx_{2n+2})+d(Ax_{2n+2},Tx_{2n+1})]}{d(Bx_{2n+1},Ax_{2n+2})+d(Sx_{2n+2},Bx_{2n+1})},\\ \dfrac{d(Tx_{2n+1},Sx_{2n+2})[d(Bx_{2n+1},Ax_{2n+2})+d(Sx_{2n+2},Bx_{2n+1})]}{d(Tx_{2n+1},Sx_{2n+2})+d(Ax_{2n+2},Tx_{2n+1})},\\ \dfrac{d(Tx_{2n+1},Bx_{2n+1})d(Tx_{2n+1},Sx_{2n+2})[d(Ax_{2n+2},Sx_{2n+2})+d(Ax_{2n+2},Bx_{2n+1})]}{d(Sx_{2n+2},Tx_{2n+1})+d(Tx_{2n+1},Bx_{2n+1})},\\ \dfrac{d(Ax_{2n+2},Sx_{2n+2})[d(Bx_{2n+1},Tx_{2n+1})+d(Bx_{2n+1},Ax_{2n+2})]}{d(Ax_{2n+2},Sx_{2n+2})+d(Tx_{2n+1},Sx_{2n+2})}\end{array}\right\}\right\}^{\lambda}$$

$$\leq \left\{ \phi \left\{ \begin{array}{c} \frac{d(y_{2n+1},y_{2n})[d(y_{2n+1},y_{2n+2})+d(y_{2n+1},y_{2n+2})]}{d(y_{2n},y_{2n+1})+d(y_{2n},y_{2n+1})}, \\ \frac{d(y_{2n+1},y_{2n})[d(y_{2n+1},y_{2n+2})+d(y_{2n+1},y_{2n+1})]}{d(y_{2n},y_{2n+1})+d(y_{2n+2},y_{2n})}, \\ \frac{d(y_{2n+1},y_{2n+2})[d(y_{2n},y_{2n+1})+d(y_{2n+2},y_{2n})]}{d(y_{2n+1},y_{2n+2})+d(y_{2n+1},y_{2n+1})}, \\ \frac{d(y_{2n+1},y_{2n})d(y_{2n+1},y_{2n+2})[d(y_{2n+1},y_{2n+2})+d(y_{2n+1},y_{2n}))]}{d(y_{2n+2},y_{2n+1})+d(y_{2n+1},y_{2n})}, \\ \frac{d(y_{2n+1},y_{2n+2})[d(y_{2n},y_{2n+1})+d(y_{2n},y_{2n+1})]}{d(y_{2n+1},y_{2n+2})+d(y_{2n+1},y_{2n+2})} \end{array} \right\} \right\}^{\lambda}$$

[using equation (2.5)]

$$\leq \left\{ \phi \left\{ \begin{array}{c} d(y_{2n+1},y_{2n+2}), \\ 1, \\ d(y_{2n+1},y_{2n+2}).\, d(y_{2n+1},y_{2n}), \\ d(y_{2n+1},y_{2n})d(y_{2n+1},y_{2n+2}), \\ d(y_{2n},y_{2n+1}) \end{array} \right\} \right\}^{\lambda}$$

$d(y_{2n+1},y_{2n+2}) \leq d^{\lambda}(y_{2n},y_{2n+1}).\, d^{\lambda}(y_{2n+1},y_{2n+2})$ [using symmetry and (ii)]

This implies that,

$d(y_{2n+1},y_{2n+2}) \leq d^{\frac{\lambda}{1-\lambda}}(y_{2n},y_{2n+1})$

On substituting, $h = \frac{\lambda}{1-\lambda} < 1$, since $\lambda \in (0,\frac{1}{2})$

(2.7) $d(y_{2n+1},y_{2n+2}) \leq d^{h}(y_{2n},y_{2n+1})$

Hence, using equations (2.6) and (2.7), we have

$d(y_n,y_{n+1}) \leq d^{h^1}(y_{n-1},y_n)$

$\leq d^{h^2}(y_{n-2},y_{n-1})$

....

....

$\leq d^{h^n}(y_0,y_1)$

for all n≥2, let m,n $\in \mathbb{N}$ such that m≥ n. Using the triangular multiplicative inequality, we obtain

$$d(y_m, y_n) \leq d(y_m, y_{m-1}).\, d(y_{m-1}, y_{m-2}) \ldots .\, d(y_{n+1}, y_n)$$

$$\leq d^{h^{m-1}}(y_1, y_0).\, d^{h^{m-2}}(y_1, y_0). \ldots d^{h^n}(y_1, y_0)$$

$$\leq d^{\frac{h^n}{1-h}}(y_1, y_0)$$

This implies that $d(y_m, y_n)$approaches to 1 as n and m approaches to infinity.

Therefore, $\{y_n\}$ is a multiplicative cauchy sequence in X.

Now, suppose that AX is complete, there exist u∈ AX such that

(2.8) $y_{n+1} = T\,x_{2n+1} = A\,x_{2n+2} \rightarrow u \quad (\text{as } n \rightarrow \infty)$.

Consequently, we can find $v \in X$ such that

(2.9) Av = u.

Further a multiplicative cauchy sequence $\{y_n\}$ has a convergent subsequence $\{y_{2n+1}\}$, therefore the sequence $\{y_n\}$ converges and hence a subsequence $\{y_{2n}\}$ also converges. Thus we have,

(2.10) $y_{2n} = S\,x_{2n} = B\,x_{2n+1} \rightarrow u \quad (\text{as } n \rightarrow \infty)$.

We claim that Sv = u, if possible Sv≠ u, substituting x = v and y = x_{2n+1} in equation (2.2), we have

$$d(Sv, Tx_{2n+1}) \leq \left\{ \phi \left\{ \begin{array}{c} \frac{d(Av, Bx_{2n+1})[d(Av, Sv) + d(Tx_{2n+1}, Sv)]}{d(Bx_{2n+1}, Tx_{2n+1}) + d(Bx_{2n+1}, Av)}, \\ \frac{d(Av, Bx_{2n+1})[d(Tx_{2n+1}, Sv) + d(Av, Tx_{2n+1})]}{d(Bx_{2n+1}, Av) + d(Sv, Bx_{2n+1})}, \\ \frac{d(Tx_{2n+1}, Sv)[d(Bx_{2n+1}, Av) + d(Sv, Bx_{2n+1})]}{d(Tx_{2n+1}, Sv) + d(Av, Tx_{2n+1})}, \\ \frac{d(Tx_{2n+1}, Bx_{2n+1})d(Tx_{2n+1}, Sv)[d(Av, Sv) + d(Av, Bx_{2n+1})]}{d(Sv, Tx_{2n+1}) + d(Tx_{2n+1}, Bx_{2n+1})}, \\ \frac{d(Av, Sv)[d(Bx_{2n+1}, Tx_{2n+1}) + d(Bx_{2n+1}, Av)]}{d(Av, Sv) + d(Tx_{2n+1}, Sv)} \end{array} \right\} \right\}^{\lambda}$$

Taking n → ∞ and using equations (2.8) to (2.10), we have

$$d(Sv,u) \leq \left\{ \phi \left\{ \begin{array}{c} \frac{d(u,u)[d(u,Sv)+d(u,Sv)]}{d(u,u)+d(u,u)}, \\ \frac{d(u,u)[d(u,Sv)+d(u,u)]}{d(u,u)+d(Sv,u)}, \\ \frac{d(u,Sv)[d(u,u)+d(Sv,u)]}{d(u,Sv)+d(u,u)}, \\ \frac{d(u,u)d(u,Sv)[d(u,Sv)+d(u,u)]}{d(Sv,u)+d(u,u)}, \\ \frac{d(u,Sv)[d(u,u)+d(u,u)]}{d(u,Sv)+d(u,Sv)} \end{array} \right\} \right\}^{\lambda}$$

$$\leq \left\{ \phi \left\{ \begin{array}{c} d(u,Sv), \\ 1, \\ d(u,Sv), \\ d(u,Sv), \\ 1 \end{array} \right\} \right\}^{\lambda}$$

$$d(u,Sv) \leq d^{\lambda}(u,Sv) \qquad \text{[using (iv)]}$$

A contradiction, since $\lambda \in (0,\frac{1}{2})$

hence, implies

(2.11) $Sv = u$.

Since $u = Sv \in SX \subset BX$, there exist $w \in X$ such that

(2.12) $u = Bw$.

Now, we claim that $Tw = u$, if possible $Tw \neq u$.

Substituting $x = v$ and $y = w$ in equation (2.2) and using equation (2.11), we have

$d(u,Tw) = d(Sv,Tw)$

$$\leq \left\{ \phi \left\{ \begin{array}{c} \frac{d(Av,Bw)[d(Av,Sv)+d(Tw,Sv)]}{d(Bw,Tw)+d(Bw,Av)}, \frac{d(Av,Bw)[d(Tw,Sv)+d(Av,Tw)]}{d(Bw,Av)+d(Sv,Bw)}, \\ \frac{d(Tw,Sv)[d(Bw,Av)+d(Sv,Bw)]}{d(Tw,Sv)+d(Av,Tw)}, \\ \frac{d(Tw,Bw)d(Tw,Sv)[d(Av,Sv)+d(Av,Bw)]}{d(Sv,Tw)+d(Tw,Bw)}, \\ \frac{d(Av,Sv)[d(Bw,Tw)+d(Bw,Av)]}{d(Av,Sv)+d(Tw,Sv)} \end{array} \right\} \right\}^{\lambda}$$

$$\leq \left\{ \phi \left\{ \begin{array}{c} \frac{d(u,u)[d(u,u)+d(Tw,u)]}{d(u,Tw)+d(u,u)}, \frac{d(u,u)[d(Tw,u)+d(u,Tw)]}{d(u,u)+d(u,u)} \\ \frac{d(Tw,u)[d(u,u)+d(u,u)]}{d(Tw,u)+d(u,Tw)}, \\ \frac{d(Tw,u)d(Tw,u)[d(u,u)+d(u,u)]}{d(u,Tw)+d(Tw,u)}, \\ \frac{d(u,u)[d(u,Tw)+d(u,u)]}{d(u,u)+d(Tw,u)} \end{array} \right\} \right\}^{\lambda}$$

[since Av = Sv = Bw = u]

$$\leq \{\phi\{1, d(Tw,u), 1, d(Tw,u), 1\}\}^{\lambda}$$

$d(u,Tw) \leq d^{\lambda}(u,Tw)$ [using (iii)]

A contradiction, since $\lambda \in (0, \frac{1}{2})$ implies

(2.13) u = Tw.

Hence, we get u = Av = Sv, i.e., v is a coincidence point of A, S.

Also u = Bw = Tw, i.e., w is coincidence point of B and T. Therefore

(2.14) Av = Sv = Bw = Tw = u.

Since the pairs (A, S) and (B, T) are weakly compatible, we have

Su = S(Av) = A(Sv) = Au = w_1 (say) and

Tu = T(Bw) = B(Tw) = Bu = w_2 (say)

From equation (2.2), we have

$d(w_1, w_2) = d(Su,Tu)$

$$\leq \left\{ \phi \left\{ \begin{matrix} \dfrac{d(Au,Bu)[d(Au,Su)+d(Tu,Su)]}{d(Bu,Tu)+d(Bu,Au)}, \dfrac{d(Au,Bu)[d(Tu,Su)+d(Au,Tu)]}{d(Bu,Au)+d(Su,Bu)}, \\ \dfrac{d(Tu,Su)[d(Bu,Au)+d(Su,Bu)]}{d(Tu,Su)+d(Au,Tu)}, \\ \dfrac{d(Tu,Bu)d(Tu,Su)[d(Au,Su)+d(Au,Bu)]}{d(Su,Tu)+d(Tu,Bu)}, \\ \dfrac{d(Au,Su)[d(Bu,Tu)+d(Bu,Au)]}{d(Au,Su)+d(Tu,Su)} \end{matrix} \right\} \right\}^{\lambda}$$

Using symmetry and above conditions of w_1 and w_2, we have

$d(w_1, w_2)$

$$\leq \left\{ \phi \left\{ \begin{matrix} \dfrac{d(w_1,w_2)[d(w_1,w_1)+d(w_2,w_1)]}{d(w_2,w_2)+d(w_2,w_1)}, \dfrac{d(w_1,w_2)[d(w_2,w_1)+d(w_1,w_2)]}{d(w_2,w_1)+d(w_1,w_2)}, \\ \dfrac{d(w_2,w_1)[d(w_2,w_1)+d(w_1,w_2)]}{d(w_2,w_1)+d(w_1,w_2)}, \\ \dfrac{d(w_2,w_2)d(w_2,w_1)[d(w_1,w_1)+d(w_1,w_2)]}{d(w_1,w_2)+d(w_2,w_2)}, \\ \dfrac{d(w_1,w_1)[d(w_2,w_2)+d(w_2,w_1)]}{d(w_1,w_1)+d(w_2,w_1)} \end{matrix} \right\} \right\}^{\lambda}$$

$$\leq \{\phi\{d(w_1,w_2), d(w_1,w_2), d(w_1,w_2), d(w_2,w_1), 1\}\}^{\lambda}$$

$d(w_1,w_2) \leq d^{\lambda}(w_1,w_2)$ [using (v)]

on the other hand, since $\lambda \in \left(0, \frac{1}{2}\right)$ implies, $d(w_1,w_2) = 1$, which implies that $w_1 = w_2$ Hence, we have

(2.15) Su = Au = Tu = Bu.

Again using equation (2.2) and symmetry of multiplicative metric space, we have

d(Sv,Tu)

$$\leq \left\{ \phi \left\{ \begin{array}{c} \frac{d(Av, Bu)[d(Av, Sv) + d(Tu, Sv)]}{d(Bu, Tu) + d(Bu, Av)}, \frac{d(Av, Bu)[d(Tu, Sv) + d(Av, Tu)]}{d(Bu, Av) + d(Sv, Bu)}, \\ \frac{d(Tu, Sv)[d(Bu, Av) + d(Sv, Bu)]}{d(Tu, Sv) + d(Av, Tu)}, \\ \frac{d(Tu, Bu)d(Tu, Sv)[d(Av, Sv) + d(Av, Bu)]}{d(Sv, Tu) + d(Tu, Bu)}, \\ \frac{d(Av, Sv)[d(Bu, Tu) + d(Bu, Av)]}{d(Av, Sv) + d(Tu, Sv)} \end{array} \right\} \right\}^{\lambda}$$

$$\leq \left\{ \phi \left\{ \begin{array}{c} \frac{d(Sv, Tu)[d(Sv, Sv) + d(Tu, Sv)]}{d(Tu, Tu) + d(Tu, Sv)}, \frac{d(Sv, Tu)[d(Tu, Sv) + d(Sv, Tu)]}{d(Tu, Sv) + d(Sv, Tu)}, \\ \frac{d(Tu, Sv)[d(Tu, Sv) + d(Sv, Tu)]}{d(Tu, Sv) + d(Sv, Tu)}, \\ \frac{d(Tu, Tu)d(Tu, Sv)[d(Sv, Sv) + d(Sv, Tu)]}{d(Sv, Tu) + d(Tu, Tu)}, \\ \frac{d(Sv, Sv)[d(Tu, Tu) + d(Tu, Sv)]}{d(Sv, Sv) + d(Tu, Sv)} \end{array} \right\} \right\}^{\lambda}$$

[since Av = Sv and Bu = Tu]

$$\leq \{\phi\{d(Sv, Tu), d(Sv, Tu), d(Sv, Tu), d(Tu, Sv), 1\}\}^{\lambda}$$

$d(Sv, Tu) \leq d^{\lambda}(Sv, Tu)$ [using v]

On the other hand, since $\lambda \in (0, \frac{1}{2})$ implies $d(Sv, Tu) = 1$ i.e., $Sv = Tu$, But Sv = u which implies that Tu = u and hence we have u = Su = Au = Tu = Bu.

Therefore u is a common fixed point of A, B, S and T. Similarly, we can complete the proof for the different case in which BX or TX or SX is complete.

Uniqueness

Let p and q are two different common fixed points of A, B, S, T then using symmetry of multiplicative metric space and using equation (2.2), we have

$d(p, q) = d(Sp, Tq)$

$$\leq \left\{ \phi \left\{ \begin{array}{c} \frac{d(Ap,Bq)[d(Ap,Sp)+d(Tq,Sp)]}{d(Bq,Tq)+d(Bq,Ap)}, \frac{d(Ap,Bq)[d(Tq,Sp)+d(Ap,Tq)]}{d(Bq,Ap)+d(Sp,Bq)}, \\ \frac{d(Tq,Sp)[d(Bq,Ap)+d(Sp,Bq)]}{d(Tq,Sp)+d(Ap,Tq)}, \\ \frac{d(Tq,Bq)d(Tq,Sp)[d(Ap,Sp)+d(Ap,Bq)]}{d(Sp,Tq)+d(Tq,Bq)}, \\ \frac{d(Ap,Sp)[d(Bq,Tq)+d(Bq,Ap)]}{d(Ap,Sp)+d(Tq,Sp)} \end{array} \right\} \right\}^{\lambda}$$

$$\leq \left\{ \phi \left\{ \begin{array}{c} \frac{d(p,q)[d(p,p)+d(q,p)]}{d(q,q)+d(q,p)}, \frac{d(p,q)[d(q,p)+d(p,q)]}{d(q,p)+d(p,q)}, \\ \frac{d(q,p)[d(q,p)+d(p,q)]}{d(q,p)+d(p,q)}, \frac{d(q,q)d(q,p)[d(p,p)+d(p,q)]}{d(p,q)+d(q,q)}, \\ \frac{d(p,p)[d(q,q)+d(q,p)]}{d(p,p)+d(q,p)} \end{array} \right\} \right\}^{\lambda}$$

$$\leq \left\{ \phi \left\{ \begin{array}{c} d(p,q), d(p,q), \\ d(p,q), d(q,p), 1 \end{array} \right\} \right\}^{\lambda}$$

$d(p,q) \leq d^{\lambda}(p,q)$ [using(v)]

A contradiction, since $\lambda \in (0,\frac{1}{2})$ implies $d(p,q) = 1$ i.e., $p = q$.

Which proves the uniqueness.

Corollary 2.2 Let A, B, S be mappings of a multiplicative metric space (X, d) into itself satisfying

(2.16) $SX \subset BX$ and $SX \subset AX$

(2.17)

$$d(Sx,Sy) \leq \left\{ \phi \left\{ \begin{array}{c} \frac{d(Ax,By)[d(Ax,Sx)+d(Sy,Sx)]}{d(By,Sy)+d(By,Ax)}, \frac{d(Ax,By)[d(Sy,Sx)+d(Ax,Sy)]}{d(By,Ax)+d(Sx,By)}, \\ \frac{d(Sy,Sx)[d(By,Ax)+d(Sx,By)]}{d(Sy,Sx)+d(Ax,Sy)}, \frac{d(Sy,By)d(Sy,Sx)[d(Ax,Sx)+d(Ax,By)]}{d(Sx,Sy)+d(Sy,By),}, \\ \frac{d(Ax,Sx)[d(By,Sy)+d(By,Ax)]}{d(Ax,Sx)+d(Sy,Sx)} \end{array} \right\} \right\}^{\lambda}$$

For all x, y ∈ X, where $\lambda \in (0, \frac{1}{2})$ and $\phi \in \Psi$;

(2.18) let us suppose that the pairs (A, S) and (B, S) are weakly compatible;

(2.19) one of the subspaces AX or BX or SX is complete

Then A, B and S have a unique common fixed point.

Proof. In theorem 2.1, if we put T = S, then we obtain the required result.

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