ON THE SUMMABILITY OF JACOBI SERIES BY (N, p_n, q_n) METHOD

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[1.1] DEFINITIONS AND NOTATIONS:

The Jacobi polynomials $P_n^{(\alpha,\beta)}(x), \alpha > -1, \beta > -1$ are defined by

$$2^{\alpha+\beta}(1-2xt+t^2)^{-1/2}[(1-t+(1-2xt+t^2)^{1/2}]^{-\alpha} \times\times [(1+t+(1-2xt+t^2)^{1/2}]^{-\beta}$$
$$= \sum_{n=0}^{\infty} p_n{}^{(\alpha,\beta)}(x)t^n$$

Let $f(x)$ be a function defined on the interval $-1 \leqslant x \leqslant 1$ such that the integral

$$\int_{-1}^{1} (1-x)^{\alpha}(1+x)^{\beta} f(x)dx \qquad (1.1.1)$$

exists in the sense of Lebesgue. The Fourier-Jacobi series corresponding to the function $f(x)$ is given by

$$f(x) \sim \sum_{n=0}^{\infty} a_n P_n^{(\alpha,\beta)}(x) \qquad (1.1.2)$$

where

$$a_n = \frac{1}{g_n}\int_{-1}^{1} (1-t)^{\alpha}(1+t)^{\beta} f(t) P_n^{(\alpha,\beta)}(t)dt$$

and

$$g_n = \frac{2^{\alpha+\beta+1}}{2n+\alpha+\beta+1} \cdot \frac{\Gamma(n+\alpha+1)}{\Gamma(n+1)} \cdot \frac{\Gamma(n+\beta+1)}{\Gamma(n+\alpha+\beta+1)}$$

The (N, p_n, q_n) Transform BORWEIN [1] of $s_n = \sum_{k=0}^{n} a_k$ is defined by

$$T_n = \sum_{k=0}^{n} \frac{p_{n-k} q_k s_k}{\theta_n}$$

where

$$\theta_n = \sum_{k=0}^{n} p_{n-k} q_k = (p*q)_n \; (p_{-1} = q_{-1} = \theta_{-1} = 0)$$

and

$\square_n \neq 0$ □□□ $n \geqslant 0.$

We shall also have the occasion

$$D_n = \sum_{k=0}^{n} \Delta p_k q_{n-k}$$

The series $\sum_{n=0}^{\infty} a_n$ or the sequence $\{s_n\}$ is said to be summable (N, p_n, q_n) to s, if $T_n \to s$ and $n \to \infty$ and is said to bo absolutely summable (N, p_n, q_n) if $\{T_n\} \in BV$ and when this happens, we shall symbolically by $\{s_n\} \in |N, p_n, q_n|$

The necessary and sufficient conditions for the regularity of $|N, p_n, q_n|$ mean are DAS [2]

$$q_k, p_{n-k} = O((p * q)_n), \square\square\, n \to \infty_{(k\,\square\square\square\square)} \qquad (1.1.3)$$

$$(|p| * |q|)_n = o(|(p * q)_n|), \square\square\, n \to \infty \qquad (1.1.4)$$

Condition $(1.1.3)$ is equivalent to the Condition $(1.1.4)$ that for all (fixed) K for which $q_k \neq 0$.

$$p_{n-k} = O((p * q)_n) \;\square\square\; n \to \infty \qquad (1.1.5)$$

but (1.1.5) need not hold for these values of k (if any) for which $q_k = 0.$

[1.2] INTRODUCTION : -

In 1946 Hardy and Rogosinki [5] proved the following theorem of convergence criterion for the Fourier. series of $f(t)$ at a given point $t = x$

Theorem A: If

$$\square(t) = O\{1/log\ |1/t|\} \quad (t \to O) \qquad (1.2.1)$$

and

$$A_n(x) = O\left(n^{-\delta}\right) \qquad (1.2.2)$$

for some $0 < \delta < 1$, then the Fourier series of $f(t)$ converges to s at $t = x$.

Later on. the same authors improved the first condition of the above theorem to

$$\square(t) = O\{1/log\ |1/t|\} \quad (t \to 0) \qquad (1.2.3)$$

In 1943, Iyengar [6] showed that the condition $(1.2.1)$ alone suffices to ensure the Harmonic summability (H) of the Fourier series of $f(t)$

$f(t)\ \square\square\ t = x \ \rightarrow\ s$

Later on Siddiqui [12] generalized Iyengar's theorem in the following manner.

Theorem B: If $(1.2.3)$ is satisfied, then the Fourier series of $f(t)$ is summable (H) at $t = x$.

Recently, Pati [8] has developed Iyengar's result by proving the following theorem.

Theorem C: Let (N, p_n), be a regular Nörlund method.

Let $\{p_n\}$ be a non-negative and monotonic non increasing sequence of real numbers such that $P_n \rightarrow \infty$.

If

$$log\ n = O(P_n),\ \square\square\ n \longrightarrow \infty \qquad (1.2.4)$$

and

$$\square(t) = O\left\{\frac{t}{P_r}\right\},\ \square\square\ t \longrightarrow +0 \qquad (1.2.5)$$

where $\square = \left[t^{-1}\right]$, then the Fourier series of $f(t)$ is summable (N, p_n) to sat $t = x$.

Later on Hesiang [4] generalized Pati's theorem to a further step. He proved the following theorem:

Theorem D: let (N, p_n) be a regular Nörlund method defined by a non-negative and monotonic non-increasing sequence of real numbers such that $P_n \longrightarrow \infty$ as $n \longrightarrow \infty$, and let $\square(t)$ be a positive monotonic increasing function $\square(n+1) \geqslant \psi(n)$.

If

$$\square(n) log\ n = o(P_n),\ \square\square\ n \longrightarrow \infty \qquad (1.2.6)$$

and

$$\square(t) = O\left\{\frac{\psi\{\square]\}t)}{P_r}\right\},\ \square\square\ t \rightarrow 0 \qquad (1.2.7)$$

then the Fourier series of $f(t)$ is summable (N, p_n) to s at $t = x$

The case $\square$ being a constant is Pati's theorem

Recently. Sharma [11] has established a theorem generalising the Pati's result on the Nörlund summability of Fourier Jacobi series. This result is analogous of the result of Hsiang [4] for trigonometric Fourier series.

The object of this paper is to generalise the above theorem for Fourier Jacobi series. This theorem is a generalisation of the theorem of Prasad and Saxena [9]. However, our theorem is as follows:-

Theorem; If

$$F_1(t) = \int_0^t |F(\phi)| d\phi = O\left(\frac{\psi(\tau) t^{2\alpha+2}}{\theta(R_r)}\right) as\ t \to \infty \qquad (1.2.8)$$

where

$$F(\phi) = [f(\cos\ \phi) - A](\sin\ \phi/2)^{2\alpha+1} \times (\cos\ \phi/2)^{2\beta+1}$$

and $\square(t)$ and $\square(t)$ are non-negative monotonic increasing functions of t such that

$$\square(n) \log\log n = O(\theta(R_n))\ \square\square\ n \to \infty \qquad (1.2.9)$$

$$n^{(2\alpha+1)/2} = O(R_n)\ \square\square\ n \to \infty \qquad (1.2.10)$$

and

$$\sum^{n} \frac{R_k}{k^{2\alpha+1/2} \log\ k} = O\left(R_n / n^{(2\alpha+2)/2}\right) \qquad (1.2.11)$$

as $n \to \infty$

then the series $(1.1.2)$ is summable (N, p_n, q_n) at the point $x = +1$ to sum A, provided that the condition $-1/2 \leqslant \alpha < 1/2$, $\square > -1/2$ and the antipole condition

$$\int_{-1}^{b} (1 + x)^{\frac{2\beta-3}{4}} |f(x)| dx < \infty \qquad (1.2.12)$$

are satisfied, where b is fixed and (N, p_n, q_n) is regular Nörlund method defined by the real non-negative and nan -increasing sequence $\{D_n\}$ such that

$$D_n \longrightarrow \infty\ \square\square\ n \longrightarrow \infty$$

[1.3] Lemmas: We require the following lemmas of Gupta [3] for the proof of our theorem

Lemma 1: let

$$N_n(\phi) = 2^{2+\beta} / R_n \times \sum_{k=0}^{n} D_k \lambda_{n-k} P_{n-k}^{(\alpha+1,\beta)} \times (\cos\ \phi)$$

where

$$R_n = \frac{2^{-\alpha-\beta-1}\Gamma(n+\alpha+\beta+2)}{\Gamma(\alpha+1)\Gamma(n+\beta+1)} = \frac{2^{-\alpha-\beta-1}}{\Gamma(\alpha+1)} \cdot n^{\alpha+1}$$

then (i) for $0 \leqslant \phi \leqslant 1/n$

$$|N_n(\phi)| = O(n^{2\alpha+2}) \qquad (1\cdot3\cdot1)$$

(ii) for $\frac{1}{n} \leqslant \phi \leqslant \pi - \frac{1}{n}, \alpha \geqslant -\frac{1}{2}$

$$|N_n(\phi)| = \frac{1}{R_n} O\left(\frac{n^{(2\alpha+1)/2} R_{(1/\phi)}}{\sin(\phi/2)^{(2\alpha+3)/2} \times (\cos\phi/2)^{(2\beta+1)/2}}\right) +$$

$$+O\left(\frac{n^{\frac{2\alpha-1}{2}}}{(\sin\sin\phi/2^{(2\alpha+5)/2} \times \cos\phi/2)^{(2\beta+3)/2})}\right) \qquad (1.3.2)$$

(iii) For

$$\pi - \frac{1}{n} \leqslant \phi \leqslant \pi, \alpha \geqslant -\frac{1}{2}, \beta > -\frac{1}{2}$$

$$|N_n(\phi)| = O\left(n^{(\alpha+\beta+1)}\right) \qquad (1.3\cdot3)$$

Lemma 2: The antipole condition

$$\int_{-1}^{b} (1+x)^{(2\beta-3)/4} |f(x)| dx < \infty$$

$$means \int_{a=cos^{-1}}^{\pi} \cos\cos t/2^{(2\beta-1)/2} \, |f(\cos\cos t) - A| dt < \infty \qquad (1.3.4)$$

which is further

$$\int_0^{1/n} t^{(2\beta-1)/2} |f(-\cos\cos t) - A| dt = 0(1), as\ n \to \infty \qquad (1.3.5)$$

[1.4] Proof of the theorem: Following the lines of Oberechkoff [7] the n^{th} partial sum of the series (1.1.2) at the point $x = +1$ is given by

$$s_n(1) = 2^{\alpha+\beta} \int (\sin\sin\phi/2)^{2\alpha} \left(\cos\cos\frac{\phi}{2}\right)^{2\beta} \times$$

$$\times f(\cos\cos\phi)(1, \cos\cos\phi) \sin\phi d\phi \qquad (1.4.1)$$

where $S_n(1, \cos\phi)$ denotes the nth patial Sum of the series

$$\sum_m P_m^{(\alpha,\beta)}(1) P_m^{(\alpha,\beta)}(\cos\phi)/g_m$$

where

$$g_m = \frac{2^{\alpha+\beta+1}\Gamma(m+\alpha+1)\Gamma(\alpha+\beta+1)}{(2m+\alpha+\beta+1)\Gamma(m+1)\Gamma(m+\alpha+\beta+1)}$$

RAO [10] has shown that

$$(1, cos\, cos\, \square) = \lambda_n P_n^{(\alpha+1,\beta)}(cos\, cos\, \square)$$

where

$$\square_n = \frac{2^{-\alpha-\beta-1}\Gamma(n+\alpha+\beta+2)}{\Gamma(\alpha+1)\Gamma(n+\beta+1)}$$

$$= \frac{2^{-\alpha-\beta-1}}{\Gamma(\alpha+1)} n^{\alpha+1}$$

therefore

$$S_n(1) - A = 2^{\alpha+\beta+1}\lambda_n \int_0^\pi (sin\ \phi/2)^{2\alpha+1} \times (cos\ \phi/2)^{2\beta+1} \times$$

$$\times [f(\square\square\square\, cos\ \phi) - A] P_n^{(\alpha+1,\beta)}(cos\phi)d\phi \qquad (1.4.2)$$

$$= 2^{\alpha+\beta+1}\lambda_n \int_0^\pi F(\phi) P_n^{(\alpha+1,\beta)}(cos\, cos\ \phi)d\phi$$

The Nörlund means of series $(1.1.2)$ at The point $x = +1$ is

$$t_n = \frac{1}{R_n}\sum_{k=0}^{n} D_k S_{n-k}(1)$$

$$t_{n-A} = \frac{1}{R_n}\sum_{k=0}^{n} D_k (S_{n-k}(1) - A)$$

$$= \frac{1}{R_n}\sum_{k=0}^{n} D_k 2^{\alpha+\beta+1}\lambda_{n-k} \times \int_0^\pi F(\phi) P_{n-k}^{(\alpha+1,\beta)}\phi))d\phi$$

$$= \int_0^\pi F(\phi) N_n(\phi)d\phi \qquad (1.4.3)$$

To prove our theorem we have to show that

$$I = \int_0^\pi F(\phi) N_n(\phi)d\phi$$

$$= O(1)\ \square\square\ n \to \infty.$$

We write

$$I = \left(\int_0^{\frac{1}{n}} + \int_{\frac{1}{n}}^{\delta} + \int_{\delta}^{\pi-\frac{1}{n}} + \int_{\pi-\frac{1}{n}}^{\pi}\right)^{\pi} F(\phi)N_n(\phi)d\phi$$

(where $\square$ is an adjusted constant)

$I = I_1 + I_2 + I_4 + I_4 \ (say) \qquad (1.4.4)$

Applying $(1.3.1)$ we have

$|I_1| = O(n^{2\alpha+2}\,(O(\psi(n)/\theta(R_n))\cdot n^{-2\alpha-2})$

$= \frac{O(\psi(n))}{(\theta(R_n))}$

$= O(1)$ as $n \to \infty$ by the hypothesis (1.2.9), (1.4.5).

Again by the application of $(1.3.2)$

$$|I_2| = O\left(\frac{\int_{1/n}^{\delta} \ |F(\phi)|n^{(2\alpha+1)/2}}{R_n}\right)R_{\left[\frac{1}{\phi}\right]}\left(sin\ sin\ \frac{▯}{2}\right)^{\frac{-2\alpha-3}{2}}\ d\phi +$$

$$+\,O(\int_{1/n}^{\delta} \ |\ F(\phi)|n^{(2\alpha-1)/2}(sin\ \phi/2)^{(-2\alpha-5)/2}d\phi)$$

$= I_{2.1} + I_{2.2}$ (▯▯▯) $\qquad (1.4.6)$

Now

$$|I_2| = \frac{n^{(2\alpha+1)/2}}{R_n}\int_{1/n}^{\delta} \ |F(\phi)|R_{\left[\frac{1}{\phi}\right]}/\phi^{(2\alpha+3)/2}\,d\phi$$

$$= O\left(\frac{n^{\frac{2\alpha+1}{2}}}{R_n}\right)^{\frac{1}{n}}[O\,\frac{(▯\left(\left[\frac{1}{\phi}\right]\right)\phi^{2\alpha+2}}{\theta\left(P_{\left[\frac{1}{\phi}\right]}\right)}\frac{R_{\left[\frac{1}{\phi}\right]}}{\frac{\phi(2\alpha+3)}{2}}]^{\delta} +$$

$$+O\left(\frac{n^{\frac{2\alpha+1}{2}}}{R_n}\right)^{1/n}\int_{\frac{1}{n}}^{\delta} \ O\,\frac{(\psi(1/\phi))\phi^{2\alpha+2}}{\theta(P_{\left[\frac{1}{\phi}\right]})^{1/n}}\frac{d}{d\phi}\frac{R_{\left[\frac{1}{\phi}\right]}}{\frac{\phi(2\alpha+3)}{2}}d\phi$$

$= I_{2.1.1} + I_{2.2.2} \quad (say) \qquad (1.4.7)$

But,

$$I_{2.1.1} = O(n^{(2\alpha+1)/2}[O(\psi([1/\phi])\frac{\phi^{2\alpha+2}}{\theta\left(R_{\{\left[\frac{1}{\phi}\right]\}}\right)_{\frac{1}{n}}}{}^{\delta}R_{\left[\frac{1}{\phi}\right]}]$$

$$= O\left(\frac{n^{\frac{2\alpha+1}{2}}}{R_n}\right) + \left(\frac{\psi(n)}{\theta(R_n)}\right)$$

$= O(1)$ as $n \to \infty \qquad (1.4.8)$

Now,

$$I_{2.1.2} = O\left(\frac{n^{\frac{2\alpha+1}{2}}}{R_n}\right)\int_{\frac{1}{n}}^{\delta} O\left(\psi\left(\left[\frac{1}{\phi}\right]\right)\frac{\phi^{2\alpha+2}}{\theta\left(P_{\left[\frac{1}{\phi}\right]}\right)\frac{d}{d\phi}}\right)\left(\frac{R_{\left[\frac{1}{\phi}\right]}}{\phi^{\frac{2\alpha+3}{2}}}\right)d\phi$$

$$= O\left(\frac{n^{\frac{2\alpha+1}{2}}}{R_n}\right)\int_{\frac{1}{\delta}}^{n} \frac{\psi[(x)]}{\theta(R(x))x^{-2\alpha-2}\frac{d}{dx}}(R_x x)^{\frac{2\alpha+3}{2}}dx$$

$$= O\left(\frac{n^{\frac{2\alpha+1}{2}}}{R_n}\right)+O\left(\frac{n^{\frac{2\alpha+1}{2}}}{R_n}\right)\sum_{k=a}^{n} \frac{\psi(K)}{\theta(R_k)k^{-2\alpha-2}}\ \Delta\left(R_k K^{\frac{2\alpha+3}{2}}\right)$$

where $a = [\delta^{-1}] + 1$ and □$R_k = R_{k+1} - R_k$

$$= O\left(\frac{n^{(2\alpha+1)/2}}{R_n}\right)+O\left(\frac{n^{(2\alpha+1)}}{R_n}\right)\sum_{k=a}^{n} \frac{R_k}{\log\log k}\times\frac{1}{k^{\frac{2\alpha+1}{2}}}$$

$$= O\left(\frac{n^{\left(\frac{2\alpha+1}{2}\right)}}{R_n}\right)+O\left(\frac{n^{\left(\frac{2\alpha+1}{2}\right)}}{R_n}\right)O\left(\frac{R_n}{n^{\frac{2\alpha+1}{2}}}\right)$$ by (1.2.11)

$$= O(1) \quad as\, n \to \infty \qquad (1.4.9)$$

Now, we consider $I_{2.2}$ where

$$I_{2.2} = O\left[\int_{1/n}^{\delta} |F(\phi)|n^{(2\alpha-1)/2}(\sin\theta/2)^{(-2\alpha-5)/2}d\phi\right]$$

$$= O(n^{2\alpha-1)/2})\left[\frac{\psi([1/\theta])\theta^{\frac{2\alpha-1}{2}}}{\theta(R_{[1/\theta]})}\right]_{1/n}^{\delta} +$$

$$+O\left(n^{(2\alpha-1)/2}\right)\times\int_{1/n}^{\delta} \frac{\psi((1/\theta])□^{\frac{2\alpha-3}{2}}}{\theta\left(R_{\left[\frac{1}{\theta}\right]}\right)}d\phi$$

$$= O\left(n^{\frac{2\alpha-1}{2}}\right)+O\left(\frac{\psi(n)}{\theta(R_n)}\right)+O\left(n^{\frac{2\alpha-1}{2}}\right)\int_{1/n}^{\delta} \frac{\psi([x])x^{\frac{-2\alpha-1}{2}}}{\theta(R(x))}dx.$$

Intigrating by Parts and applying (1.2.1)

$$= O(1)\ □□\ n \to \infty\ □□□□□□□\ □ < 1/2 \qquad (1.4\cdot10)$$

Now, we consider I_3

where

$$|I_3| = O(\frac{\int \quad |F(\phi))n^{\frac{2\alpha+1}{2}}}{R_n} \cdot \frac{R_{[1/\phi]}d\phi}{(sin\ \phi/2)^{(2\alpha+\frac{\phi}{2}}} (cos\ \phi/2)^{\frac{2\beta+1}{2}})$$

$$+ O\left(n^{\frac{2\alpha-1}{2}}\right)\int_{\delta}^{\pi-\frac{1}{n}} \frac{|F(\phi)|d\phi}{(sin\ \phi/2)^{\frac{2\alpha+5}{2}}} \times (cos_{\phi/2})^{(\frac{2\beta+3}{2})}$$

$$= O\left(\frac{n^{\frac{2\alpha-1}{2}}}{R_n}\right)\int_{\delta}^{\pi-\frac{1}{n}} \quad |f(cos\ cos\ ◻) - A|\left(\frac{cos\ cos\ ◻}{2}\right)^{\frac{2\beta-1}{2}})\left(\frac{◻◻◻ cos\ ◻}{2}\right)d\phi +$$

$$+O\left(n^{\frac{2\alpha-1}{2}}\int_{\delta}^{\pi-\frac{1}{n}} \quad |f(cos\ cos\ ◻) - A|\right)\left(\frac{cos\ cos\ ◻}{2}\right))^{\frac{2\beta-1}{2}}d\phi$$

$$= O\left(\frac{n^{\frac{2\alpha+1}{2}}}{R_n}\right) + O\left(n^{\frac{2\alpha-1}{2}}\right) \qquad ◻◻◻◻\ (1.3.4)$$

$$= O(1)\ ◻◻\ n \to \infty, \alpha < \frac{1}{2} \qquad (1.4.11)$$

Lastly.

$$|I_4| = \int_{\pi-1/n}^{\pi} \quad |F(\phi)N_n(\phi)d\phi|$$

$$= O\left(n^{\alpha+\beta+1}\right)\int_{\pi-\frac{1}{n}}^{\pi} \quad |f(cos\ cos\ ◻) - A| \times$$

$$\times\ (cos\ \phi/2)^{(\frac{2\beta+1}{2})}(sin\ \phi/2)^{(2\alpha+1)}d\phi$$

$$= O\left(n^{(2\alpha-1/2)}\right)\int_0^{1/n} \quad |f(-cos\ \phi) - A||^{(\alpha\beta-1)/2}d\phi$$

$= O(1)$ by the application of $(1.3.5)$ $\qquad (1.4.12)$

Combining $(1 \cdot 4.4)$ to $(1 \cdot 4 \cdot 12)$

$$I = O(1)\ ◻◻\ n \longrightarrow \infty$$

This complete the proof of the theorem

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