THE COMBINATION OF SPIDER GRAPHS WITH STAR GRAPHS FORMS GRACEFUL

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Abstract: A labeled graph $G$ which can be gracefully numbered is said to be graceful. Labeling the node of $G$ with distinct non negative integers and then labeling the edges of $G$ with the absolute differences between nodes values, if the graph edges numbers runs from 1 to $n$, the graph $G$ is gracefully numbered. In this paper, we combine some spider graphs with star graphs forming new graphs and show them to be graceful, which have a wide range of applications in different fields such as coding theory, X-ray crystallography, radar, astronomy, circuit design, communication network and data base management systems.

Keywords: Graceful, Graph labeling, Spider, Star.

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1. INTRODUCTION

Graph Theory is a branch of discrete mathematics, distinguished by geometric approach to the study of objects. The principle object of the theory is a graph and its generation. Any problem or object under consideration is represented in the form of nodes (Vertices, points, elements) and edges (lines, link, connections). The vertex set and edge set are denoted by \( V(G) \) and \( E(G) \) respectively [1]. The Ringel - Kotzig conjectured that all trees are graceful and this has been the focus of many papers [2].

A graceful labeling [3] or numbered \( f \) of a graph \( G \) with \( q \) edges is \( f: G \to \{0, 1, 2, \ldots, q\} \) is an injective function such that when each edge \( xy \) is assigned the label \( |f(x) − f(y)| \) and also the resulting labels are distinct and non-zero. C.Huang called the effort to prove it a “disease” Among the trees known to be graceful is: Caterpillars (A caterpillar is a tree with a property that the removal of its endpoints leaves the path [5] trees with at most 4 end - vertices [5][6][7].

J.C.Bermond conjectured [8] that lobsters are graceful (A lobster is a tree with the property that the removal of the endpoints leaves a caterpillar) Symmetrical trees (A symmetrical tree is a rooted tree in which every level contains vertices of the same degree [8][9]. Regular bamboo trees (A rooted tree consisting of branches of equal length the end points of which are identified with end points of stars of equal size [10], are graceful. Olive trees [11] (A rooted tree consisting of \( k \) branches where the \( i \)th branch is a path of length \( i \)), are graceful.

K.Eshghi & P.Azimi [12][13] discussed a programming model for finding graceful labelings of large graphs. The computational results showed that the model can easily solve the graceful labeling problem for large graphs. They used this method to verify that all trees with 30, 35 and 40 vertices are graceful.

D.Morgan investigated that all lobsters with perfect matching are graceful [14][15][16].

\textit{Spider graphs} \( S_{m,n} \) (\( m \geq 1, n \geq 1 \)).

\textit{Spider graphs} \( R_{E_{m,q}} \) (\( m \geq 1, n \geq 1 \) have \( q \) legs of length 2 and \( m \) legs of length 1.

In this paper we will form graceful graphs by combining stars with the outer vertices of spider graphs.
1.1. Preliminary assumptions.

- All graphs considered here are finite simple and undirected.
- The vertex set and edge set are denoted by \( V(G) \) and \( E(G) \). Our notations and terminology are as in \(^{[1]}\)
- We refer\(^{[4]}\) for some basic concepts.

1.2. Basic Definitions

They are needed for further discussion.

**Definition - 1:**
A graph \( G = (V,E) \) consists of a finite set denoted by \( V \) or \( V(G) \), and a collection \( E, E(G) \) of unordered pairs \( u,v \) of distinct elements from \( V \). Each elements of \( V \) is called a vertex or a point or a node or element and each element of \( E \) is called an edge or a line or a link or a connection. Usually these graphs are also called simple graphs.

**Definition - 2:**
The number of vertices namely the cardinality of \( V \) is called the order of \( G \) and is denoted by \( |V| \) or \( n \) or \( p \)

**Definition - 3:**
The number of edges of a graph namely the cardinality of \( E \) is called the size of \( G \), and is denoted by \( |E| \) or \( m \) or \( q \), we write \( e = v_i v_j \in E(G) \) to mean that the pair \( v_i, v_j \in E(G) \) when \( e = v_i v_j \in E(G) \)

We say that \( v_i \) and \( v_j \) are adjacent, \( e \) and \( v_i, e \) and \( v_j \) are incident

**Definition - 4:**
The open neighborhood \( \mathcal{N}(v) \) of the vertex \( v \) consists of the set of vertices adjacent to \( v \). That is \( \mathcal{N}(v) = \{ w \in V : wv \in E \} \). For a set \( S \subset V \), the open neighborhood \( \mathcal{N}(S) \) is defined by \( \mathcal{N}(S) = \bigcup_{v \in S} \mathcal{N}(V) \)

The closed neighborhood of the vertex \( v \) is \( \mathcal{N}[v] = \mathcal{N}(v) \cup \{v\} \) and the closed neighborhood \( \mathcal{N}[S] \) by \( \mathcal{N}[S] = \mathcal{N}(S) \cup S \)

**Definition - 5:**
The degree of a vertex \( v \) is denoted by \( \deg(v) \), and defined as the number of edges incident with \( v \) that is \( \deg(v) = \{ N(v) \} \). The degree sequence of \( G \) is \( \{ \deg v_1, \deg v_2, ..., \deg v_n \} \), typically written in non - decreasing or non - increasing.
Definition - 6:
The maximum and minimum of the degree of vertices of a graph $G$ are denoted by $\Delta(G)$ and $\delta(G)$ respectively.
If $\Delta(G) = \delta(G) = r$ then $G$ is called a regular graph of degree $r$ or simply $r$-regular.
Note: A 3-regular graph is also called 3-regular.

Definition - 7:
A walk of length $k$ it is an alternating sequence $W = u_0e_1u_1e_2u_2 ... u_{k-1}e_ku_k$ of vertices and edges with $e_i = u_{i-1}u_i$. It is also denoted by $W = u_0u_1u_2 ... u_{k-1}u_k$ (implicitly $u_iu_{i+1} \in E$, $0 < i < k - 1$)
If all $k$ edges are distinct, then $W$ is called a trial.
If all $k + 1$ vertices are distinct, then $W$ is called a path.
If $u_0 = u_k$, then $W$ is called closed and closed trial is called a circuit.
If $u_0 = u_k$ and $u_1u_2 ... u_{k-1}$ are distinct then $W$ is called a cycle or k-cycle.
If $u_0 = u$ and $u_k = v$ then $W$ is said to be a $u - v$ walk of length $k$.

Definition - 8:
A graph is said to be connected, if every pair $u, v$ of vertices $G$ there exists a $u - v$ path. Otherwise $G$ is said to be disconnected.

Definition - 9:
If there exists at least one $u - v$ walk, then the distance $d(u, v)$ between $u$ and $v$, it is the minimum length of $u - v$ walk; if no $u - v$ walk exists then it is denoted by $d(u, v) = \infty$.

Definition - 10:
The $i^{th}$ open neighborhood of $v$ for $i \geq 0$ consists of those vertices at distance $i$ from $v$ and is denoted by $N^i(v)$.
That is, $N^i(v) = \{u \in V : d(u, v) = i\}$
The $i^{th}$ closed neighborhood $N^i[v] = \{u \in V : d(u, v) \leq i\}$
In particular $N^0(v) = N^0[v] = v$, $N^1(v) = N(v)$ and also $N^1[v] = N[v]$.

Definition - 11:
The eccentricity of a vertex $v$, denoted by $ecc(v)$, is defined as $ecc(v) = \max \{ecc(v) : v \in V\}$
The radius of a graph $G$, denoted by $rad(G)$, is defined as $rad(G) = \min \{ecc(v) : v \in V\}$.
The diameter of a graph $G$, denoted by $diam(G)$, is defined as
$$diam(G) = \max\{ecc(v) : v \in V\}$$
The distance of a vertex $v$, denoted by $dist(v)$, is defined as
$$dist(v) = \sum_{u \in V(G)} d(u,v)$$

**Definition - 12:**
The cycle $C_n$ of order $n \geq 3$ has size $m = n$ (number of vertices = number of edges) is connected and is 2 - regular.
A tree is a connected graph with no cycle.
The path $P_n$ is the tree of order $n$ with diameter $n - 1$
The star $K_{1,n-1}$, it has one vertex of $(n - 1)$ degree and $(n - 1)$ vertices of degree one.
In any graph, a vertex degree one is called an end vertex and an edge incident with and end vertex is called pendant edge. Among all connected graphs of order, trees have the minimum number of edges and the complete graph or clique (A graph in which any two vertices are adjacent). Any tree is a bipartite (A graph whose vertices can be divided into two disjoint sets $U$ and $V$ such that every edge has one end in $U$ and another end in $V$). A complete bipartite graph is a special kind of bipartite graph where every vertex of the first set is connected to every vertex of the second set. All complete bipartite graphs are stars when they are trees. A star $S_n$ is the complete bipartite graph $K_{1,n}$

**Definition - 13:**
Two graphs $G$ and $H$ are said to be isomorphic if there exists a bijection $\chi: V(G) \rightarrow V(H)$ such that $uv \in E(G)$, if and only if $\chi(u) \chi(v) \in E(H)$.

**1.3. Construction of a graph from two given graphs**
Let $G$ and $H$ are two given graphs then
1. **Union** $K = G \cup H$ has $V(K) = V(G) \cup V(H)$ and $E(K) = E(G) \cup E(H)$
2. **Intersection**, let $V(G) \cup V(H) = \emptyset$. The **join** $K = G + H$ has $V(K) = V(G) \cup V(H)$ and also $E(K) = E(G) \cup E(H) \cup \{uv: u \in V(G) \text{ and } v \in V(H)\}$
3. The Cartesian product $K = G \times H$ has $V(K) = V(G) \times V(H)$ and $(u_1, v_1), (u_2, v_2) \in V(K)$ are adjacent if $u_1 = u_2, v_1, v_2 \in E(H)$ and $v_1 = v_2, u_1, u_2 \in E(G)$
4. $P_j \times P_k$ is called the grid $G_{j,k}$
5. $C_j \times C_k$ is called the cylinder $C_{j,k}$
6. $C_j \times C_k (j, k \geq 3)$ is called the Torus $T_{j,k}$
2. NECESSARY DEFINITION FOR FURTHER APPROACH

1) A tree $T_n$ on $n$ vertices $v_1, v_2, v_3, ..., v_n$ can be represented by,

$$T_n = T_n(\text{deg}(v_1), \text{deg}(v_2), ..., \text{deg}(v_n)),$$

where $\text{deg}(v_n)$ is the $n$th vertex in $T_n$.

a. A tree on three vertices is denoted by $T_3 = T_3(1, 1, 2)$

b. Up to isomorphism the trees $T_4$ with four vertices are $T_4(1, 1, 1, 3)$,

$$T_4(1, 1, 2, 2)$$

c. Up to isomorphism the trees $T_5$ with five vertices are $T_5(1, 1, 1, 1, 4)$, $T_5(1, 1, 1, 2, 3)$, $T_5(1, 1, 1, 2, 4)$

d. Up to isomorphism the trees $T_6$ with six vertices are $T_6(1, 1, 1, 1, 1, 5)$, $T_6(1, 1, 1, 1, 2, 4)$, $T_6(1, 1, 1, 1, 3, 3)$, $T_6(1, 1, 2, 2, 2)$

And two different classes of trees $T_6(1, 1, 1, 2, 2, 3)$

2) $S_n$ is a star graph with $(n + 1)$ vertices and $n$ legs of length one

3) Spider graph $S_{m,n}$ is defined as a tree with $m$ legs of length $n$ on $mn + 1$ vertices

4) $RE_{m,n}$ is a tree with $m$ legs of length one and $n$ legs of length two

Let

$o$ is an outer vertex, $\alpha$ is a middle vertex, $c$ is the center vertex and $p$ is the pendent vertex attached to center

5) Let $A$ and $B$ are any two graphs. A graph $A \ast B$ is obtained by gluing a vertex $a$(of deg($d$)) of the graph $A$ with a vertex $b$ (of deg($d$))of the graph $B$.

a. If a center vertex of $A$ is attached to center vertex of $B$, then $I$ is of category $I$

b. If a non center vertex of $A$ is attached to center vertex of $B$, then $A \ast B$ is of category $Ii$

c. If a center vertex of $A$ is attached to non center vertex of $B$, then $A \ast B$ is of category $III$

d. If a non center vertex of $A$ is attached to non center vertex of $B$, then $A \ast B$ is of category $IV$

6) Definition - Category $I$

If a center vertex of $A$ of any degree is attached to center vertex of $B$, of any degree then $A \ast B$ is called $A \ast B$ of category $I$, type $I$. Up to isomorphism, $A \ast B$ has only one type in category $I$
7) Definition - Category II
   a. If a non center vertex of $A$ of degree one is attached to center vertex of $B$, of any degree then $A * B$ is called $A * B$ of category II, type I.
   b. If a non center vertex of $A$ of degree is two attached to center vertex of $B$, of any degree then $A * B$ is called $A * B$ of category II, type II.
   c. Generally, if a non center vertex of $A$ of degree is $m$ attached to center vertex of $B$, of any degree then $A * B$ is called $A * B$ of category II, type $m$.

8) Definition - Category III
   a. If a center vertex of $A$ of any degree is attached to non center vertex of $B$, of degree one then $A * B$ is called $A * B$ of category III, type I.
   b. If a center vertex of $A$ of any degree is attached to non center vertex of $B$, of degree two then $A * B$ is called $A * B$ of category III, type II.
   c. Generally, if a center vertex of $A$ of any degree is attached to non center vertex of $B$, of degree $m$ then $A * B$ is called $A * B$ of category III, type $m$.

9) Definition - Category IV
   a. If a non center vertex of $A$ of degree one is attached to non center vertex of $B$, of degree one then $A * B$ is called $A * B$ of category IV, type I.
   b. If a non center vertex of $A$ of degree two is attached to non center vertex of $B$, of degree one then $A * B$ is called $A * B$ of category IV, type II.
   c. Generally, if a non center vertex of $A$ of degree $m$ is attached to non center vertex of $B$, of degree one then $A * B$ is called $A * B$ of category IV, type $m$.

2.1. Combination of spider graphs with one copy of star graph

Basic Theorems:
The combined graph which is $S_{m,2} * S_n$ of category II, type I is graceful, for $m = 3, 4, 5$ and positive integer $n \geq 2$

Proof:
Here we will prove this theorem in three cases

Case-1: For $m = 3$
The combined graph which is $S_{3,2} * S_n$ of category II, type I is graceful where $S_{3,2}$ is a spider graph with seven vertices and $S_n$ is a star with $(n + 1)$ vertices, for $n \geq 2$
Let us take $v_1, v_2, \ldots, v_7$ are the vertices of the $S_{3,2}$ with the label $v_1$ denoting center
and \(v_2, v_4\) and \(v_7\) representing the outer vertices and \(v_3, v_5, v_6\) representing the inner vertices

Let us take \(v_7, v_8, \ldots, v_{n+7}\) are the vertices of the star \(S_n\)

Connect the center of the star \(S_n\) is with vertex \(v_7\), and the other \(n\) vertices of \(S_n\) are named,

By giving the labels \(v_8, \ldots, v_{n+7}\) continuously starting from one pendant vertex till the last pendant vertex

Now we will get a new graph and is denoted by \(S_{3,2} \ast S_n\), clearly it is a tree with

\((n + 7)\) vertices and \((n + 6)\) edges as shown in the following figure – 1:

![Diagram](attachment:diagram.png)

**Figure 1. Ordinary labeling of \(S_{3,2} \ast S_n\) of category II, type I**

The labeling \(f\) on the vertices of \(S_{3,2} \ast S_n\) is defined by

\[
\begin{align*}
    f(v_i) &= i & \text{for } i = 1, 3, 6, 7 \\
    f(v_i) &= i + 2 & \text{for } i = 2 \\
    f(v_i) &= i - 2 & \text{for } i = 4 \\
    f(v_i) &= i - 1 & \text{for } 8 \leq i \leq n + 7
\end{align*}
\]

Then, the labeling \(f\) on the vertices of \(S_{3,2} \ast S_n\) of category II, type I, satisfies the conditions for a graceful labeling

Hence complete the proof of case - 1.

**Case - 2: For \(m = 4\)**

The combined graph which is \(S_{4,2} \ast S_n\) of category II, type I is graceful,

where \(S_{4,2}\) is a spider graph with nine vertices and \(S_n\) is a star with 

\((n + 1)\) vertices, for \(n \geq 2\)
Let us take $v_1, v_2, ..., v_9$ are the vertices of the spider $S_{4,2}$ with the label $v_1$ denoting the center and $v_2, v_4, v_6$ and $v_9$ representing the outer vertices and $v_3, v_5, v_7$ representing the inner vertices.

Let us take $v_{10}, v_{11}, ..., v_{n+9}$ are the vertices of the star $S_n$.

Connect the center of the star $S_n$ is with vertex $v_9$, and the other $n$ vertices of $S_n$ are named,

By giving the labels $v_{10}, v_{11}, ..., v_{n+9}$ continuously starting from one pendant vertex till the last pendant vertex.

Now we will get a new graph and is denoted by $S_{4,2} \ast S_n$, clearly it is a tree with $(n + 9)$ vertices and $(n + 8)$ edges as shown in the following figure − 2:

![Graph Image](image-url)

Figure 2. Ordinary labeling of $S_{4,2} \ast S_n$ of category II, type I.

The labeling $f$ on the vertices of $S_{4,2} \ast S_n$ is defined by

\[
\begin{align*}
  f(v_i) &= 0 & \text{for } i = 1 \\
  f(v_i) &= i + 1 & \text{for } i = 2 \\
  f(v_i) &= i + n + 2 & \text{for } i = 3, 4, 5 \\
  f(v_i) &= 1 & \text{for } i = 6 \\
  f(v_i) &= i + n - 3 & \text{for } i = 7 \\
  f(v_i) &= i + n & \text{for } i = 8 \\
  f(v_i) &= i - 7 & \text{for } i = 9 \\
  f(v_i) &= i - 6 & \text{for } 10 \leq i \leq n + 9
  \end{align*}
\]
Then, the labeling \( f \) on the vertices of \( S_{4,2} \ast S_n \) of category II, type I, satisfies the conditions for a graceful labeling.
Hence complete the proof of case - 2.

**Case - 3: For \( m = 5 \)**

The combined which is \( S_{5,2} \ast S_n \) of category II, type I is graceful graph \( S_{5,2} \ast, \) where \( S_{5,2} \) is a spider graph with eleven vertices and \( S_n \) is a star with \((n + 1)\) vertices, for \( n \geq 2 \).

Let us take \( v_1, v_2, v_3, \ldots, v_{11} \) are the vertices of the spider \( S_{5,2} \) with the label \( v_1 \) denoting the center and \( v_2, v_4, v_6, v_8 \) and \( v_{11} \) representing the outer vertices and \( v_3, v_5, v_7, v_9, v_{11} \) representing the inner vertices.

Let us take \( v_{12}, v_{13}, \ldots, v_{n+11} \) are the vertices of the star \( S_n \).

Connect the center of the star \( S_n \) is with vertex \( v_9 \), and the other \( n \) vertices of \( S_n \) are named, By giving the labels \( v_{10}, v_{11}, \ldots, v_{n+9} \) continuously starting from one pendant vertex till the last pendant vertex

Now we will get a new graph and is denoted by \( S_{5,2} \ast S_n \), clearly it is a tree with \((n + 11)\) vertices and \((n + 10)\) edges as shown in the following figure – 3:

![Figure 3. Ordinary labeling of \( S_{5,2} \ast S_n \) of category II, type I](image)

The labeling \( f \) on the vertices of \( S_{4,2} \ast S_n \) is defined by
\[
f(v_i) = 0 \quad \text{for} \ i = 1
\]
\[ f(v_i) = i \quad \text{for } i = 2, 5, 7 \]
\[ f(v_i) = i + 3 \quad \text{for } i = 3 \]
\[ f(v_i) = i - 1 \quad \text{for } i = 4, 9 \]
\[ f(v_i) = i - 2 \quad \text{for } i = 6 \]
\[ f(v_i) = i + 1 \quad \text{for } i = 8 \]
\[ f(v_i) = i + n \quad \text{for } i = 10 \]
\[ f(v_i) = 1 \quad \text{for } i = 11 \]
\[ f(v_i) = i - 2 \quad \text{for } 12 \leq i \leq n + 11 \]

Then, the labeling \( f \) on the vertices of \( S_{5,2} \ast S_n \) of category II, type I, satisfies the conditions for a graceful labeling.

Hence complete the proof of case - 3.

2.2. Combination of general graphs \( RE_{m,q} \) of category II, type I with one copy of star graphs

**Theorem -2:**
For \( n \geq 1, m \geq 1 \) and \( q \geq 1 \), the graph \( RE_{m,q} \ast S_n \), of category II, type I, then it is graceful if \( RE_{m,q} \) is spider graph on \((m + 2q + 1)\), vertices and also \( S_n \) is a star graph with \((n + 1)\) vertices.

**Proof:**
Before we prove this theorem, we can prove the following theorems

**Theorem - 2 a:**
For \( n \geq 2, m \geq 2 \), the graph \( RE_{m,1} \ast S_n \), of category II, type I, then it is graceful if \( RE_{m,1} \) is spider graph on \((m + 3)\), vertices and also \( S_n \) is a star graph with \((n + 1)\) vertices.

**Proof:**
Let \( RE_{m,1} \) is a tree with \( m \) legs of length one and one leg with length two.
Let \( v_1, v_2, v_3, \ldots, v_m, v_{m+2} \) are the pendent vertices of the tree \( RE_{m,1} \) with the label \( v_{m+3} \) has degree \((m + 1)\) and the label \( v_{m+1} \) representing the vertex adjacent to \( v_{m+2} \) and \( v_{m+3} \) of degree 2.
The star has \((n + 1)\) vertices they have been labeled by giving the labels \( w_1, w_2, w_3, \ldots, w_{n+1} \)
Continuously starting from one pendent vertex till the last pendent vertex and take the label $w_1$ for the center of $S_n$

To form the graph $RE_{m,1} \ast S_n$, of category II, type I

First the vertex $w_1$ of $S_n$ is merged with the vertex $v_{m+2}$ of $RE_{m,1}$ and labeled as $v_{m+2}$

There are $(m + n + 3)$ vertices and $(m + n + 2)$ edges

The name of the labels $w_2, w_3, ..., w_{n+1}$ are changed as $v_{m+4}, v_{m+5}, ..., v_{m+n+3}$ respectively.

As shown in the following figure – 4:

![Figure 4. Ordinary labeling of $RE_{m,1} \ast S_n$ of category II, type I](image)

The labeling $f$ on the vertices of $RE_{m,1} \ast S_n$ is defined by

$f (v_i) = i$ for $1 \leq i \leq m$

$f (v_i) = i - m - 1$ for $i = m + 1$

$f (v_i) = i - 1$ for $i = m + 2$

$f (v_i) = i + n - 1$ for $i = m + 3$

$f (v_i) = i - 2$ for $m + 4 \leq i \leq m + n + 3$

Then, the labeling $f$ on the vertices of $RE_{m,1} \ast S_n$ of category II, type I, satisfies the conditions for a graceful labeling

Hence complete the proof of theorem -2 a.
Theorem - 2 b:

For \( n \geq 1, m \geq 1 \), the graph \( RE_{m,2} \ast S_n \), of category II, type I, then it is graceful if \( RE_{m,2} \) is spider graph on \((m + 5)\) vertices and also \( S_n \) is a star graph with \((n + 1)\) vertices.

Proof:

Let \( RE_{m,2} \) is a tree with \( m \) legs of length one and two legs with length two.

Let \( v_1 \) is the center and \( b, c \) are middle vertices and \( d, e \) are the outer vertices and \( v_2, v_3, \ldots, v_m, v_{m+1}, \ldots \) are single vertices of \( RE_{m,2} \).

The star has \((n + 1)\) vertices they have been labeled by giving the labels \( v_{m+6}, \ldots, v_{m+n+5} \)

Continuously starting from one pendent vertex till the last pendent vertex and take the label \( v \) for the center of \( S_n \). To form the graph \( RE_{m,2} \ast S_n \), of category II, type I

First the vertex \( v \) of \( S_n \) is merged with the vertex \( e \) of \( RE_{m,2} \) vertex adjacent to \( c \) and labeled as \( v_{m+5} \).

There are \((m + n + 5)\) vertices and \((m + n + 4)\) edges.

The name of the labels \( b, c \) and \( d \) are changed as \( v_{m+2}, v_{m+3}, \) and \( v_{m+4} \) respectively.

As shown in the following figure – 5

![Figure 5. Ordinary labeling of \( RE_{m,2} \ast S_n \) of category II, type I](image)
The labeling \( f \) on the vertices of \( RE_{m,2} \ast S_n \) is defined by

\[
\begin{align*}
  f(v_1) &= 1 & \text{for } i = 1 \\
  f(v_i) &= i + n + 2 & \text{for } 2 \leq i \leq m + 2 \\
  f(v_i) &= n + 3 & \text{for } i = m + 3 \\
  f(v_i) &= 0 & \text{for } i = m + 4 \\
  f(v_i) &= i - m - 3 & \text{for } m + 5 \leq i \leq m + n + 5
\end{align*}
\]

Then, the labeling \( f \) on the vertices of \( RE_{m,2} \ast S_n \) of category II, type I, satisfies the conditions for a graceful labeling.

Hence complete the proof of theorem -2 b.

**Theorem - 2 c:**

For \( n \geq 1, m \geq 1 \), the graph \( RE_{m,3} \ast S_n \), of category II, type I, then it is graceful if \( RE_{m,3} \) is spider graph on \( (m + 7) \) vertices and also \( S_n \) is a star graph with \( (n + 1) \) vertices.

**Proof:**

Let \( RE_{m,4} \) is a tree with \( m \) legs of length one and three legs with length two.

Let \( v_1 \) is the center and \( b, c, d \) are middle vertices and \( e, y, g \) are the outer vertices and

\( v_2, v_3, \ldots, v_m, v_{m+1} \), are single vertices of \( RE_{m,3} \)

The star has \((n + 1)\) vertices they have been labeled by giving the labels \( v_{m+10}, \ldots, v_{m+n+9} \)

Continuously starting from one pendent vertex till the last pendent vertex and take the label \( v \) for the center of \( S_n \)

To form the graph \( RE_{m,3} \ast S_n \), of category II, type I

First the vertex \( v \) of \( S_n \) is merged with the vertex \( g \) of \( RE_{m,3} \) adjacent to \( d \) and labeled as \( v_m + 7 \). There are \((m + n + 7)\) vertices and \((m + n + 6)\) edges.

The name of the labels \( b, c, d, e \) and \( y \) are changed as \( v_{m+2}, v_{m+3}, v_{m+4}, v_{m+5} \) and \( v_{m+6} \) respectively. As shown in the following figure – 6:
Figure 6. Ordinary labeling of $RE_{m,3} * S_n$ of category II, type I

The labeling $f$ on the vertices of $RE_{m,3} * S_n$ is defined by

- $f(v_i) = 1$ for $i = 1$
- $f(v_i) = i + 1$ for $2 \leq i \leq m + 2$
- $f(v_i) = m + n + 6$ for $i = m + 3$
- $f(v_i) = m + n + 5$ for $i = m + 4$
- $f(v_i) = m + 4$ for $i = m + 5$
- $f(v_i) = 0$ for $i = m + 6$
- $f(v_i) = 2$ for $i = m + 7$
- $f(v_i) = i - 3$ for $m + 8 \leq i \leq m + n + 7$

Then, the labeling $f$ on the vertices of $RE_{m,3} * S_n$ of category II, type I satisfies the conditions for a graceful labeling. Hence complete the proof of theorem -2 c.

**Theorem - 2 d:**

For $n \geq 1, m \geq 1$, the graph $RE_{m,A} * S_n$, of category II, type I, then it is graceful if $RE_{m,A}$ is spider graph on $(m + 9)$ vertices and also $S_n$ is a star graph with $(n + 1)$ vertices.

**Proof:**

Let $RE_{m,A}$ is a tree with $m$ legs of length one and four legs with length two.

Let $v_1$ is the center and $b, c, d, e$ are middle vertices and $y, g, h, i$ are the outer vertices and $v_2, v_3, ..., v_m, v_{m+1}$ are single vertices of $RE_{m,A}$.
The star has \((n + 1)\) vertices they have been labeled by giving the labels \(v_{m+10}, \ldots, v_{m+n+9}\). Continuously starting from one pendent vertex till the last pendent vertex and take the label \(v\) for the center of \(S_n\).

To form the graph \(RE_{m,A} * S_n\) of category II, type I,

First the vertex \(v\) of \(S_n\) is with the outer vertex \(i\) of \(RE_{m,A}\) adjacent to \(e\) and labeled \(v_{m+9}\).

There are \((m + n + 9)\) vertices and \((m + n + 8)\) edges.

The name of the labels \(b, c, d, e, y, g\) and \(h\) are changed as \(v_{m+2}, v_{m+3}, v_{m+4}\), \(v_{m+5}, v_{m+6}, v_{m+7}\) and \(v_{m+8}\) respectively.

The labeling \(f\) on the vertices of \(RE_{m,A} * S_n\) is defined by

\[
\begin{align*}
 f(v_1) &= 1 & \text{for } i = 1 \\
 f(v_1) &= 4 & \text{for } i = 2 \\
 f(v_i) &= i + n + 6 & \text{for } 3 \leq i \leq m + 1 \\
 f(v_i) &= 3 & \text{for } i = m + 2 \\
 f(v_i) &= 6 & \text{for } i = m + 3 \\
 f(v_i) &= m + n + 8 & \text{for } i = m + 4 \\
 f(v_i) &= n + 8 & \text{for } i = m + 5 \\
 f(v_i) &= 7 & \text{for } i = m + 6 \\
 f(v_i) &= 5 & \text{for } i = m + 7 \\
 f(v_i) &= 0 & \text{for } i = m + 8 \\
 f(v_i) &= 2 & \text{for } i = m + 9 \\
 f(v_i) &= i - m - 2 & \text{for } m + 10 \leq i \leq m + n + 9
\end{align*}
\]

Then, the labeling \(f\) on the vertices of \(RE_{m,A} * S_n\) of category II, type I, satisfies the conditions for a graceful labeling.

Hence complete the proof of theorem -2 d. The graph is shown in the following figure - 7.
Theorem - 2 e:
For $n \geq 1, m \geq 1$, the graph $RE_{m,5} \ast S_n$, of category II, type I, then it is graceful if $RE_{m,5}$ is a spider graph on $(m + 11)$ vertices and also $S_n$ is a star graph with $(n + 1)$ vertices.

Proof:
Let $RE_{m,5}$ is a tree with $m$ legs of length one and five legs with length two.
Let $v_1$ is the center and $b, c, d, e, y$ are middle vertices and $g, h, i, j, k$ are the outer vertices and $v_2, v_3, \ldots, v_m, v_{m+1}$, are single vertices of $RE_{m,5}$
The star $S_n$ has $(n + 1)$ vertices which is tagged by giving the labels $v_{m+12}, \ldots, v_{m+n+11}$
Continuously starting from one pendent vertex till the last pendent vertex and take the label $v$ for the center of $S_n$
To form the graph $RE_{m,5} \ast S_n$, of category II, type I
First the vertex $v$ of $S_n$ is combined with the pendent vertex $k$ of $RE_{m,5}$ which is adjacent to $y$ and labeled as $v_{m+11}$. There are $(m + n + 11)$ vertices and $(m + n + 10)$ edges.
The name of the labels $b, c, d, e, y, g, h, i$ and $j$ are changed as $v_{m+2}, v_{m+3}, v_{m+4}$, $v_{m+5}, v_{m+6}$, $v_{m+7}, v_{m+8}, v_{m+9}$ and $v_{m+10}$ respectively. As shown in the following figure – 8:

![Graph](image)

**Figure 8.** Ordinary labeling of $RE_{m,5} * S_n$ of category II, type I

The labeling $f$ on the vertices of $RE_{m,4} * S_n$ is defined by

- $f(v_i) = 1$ for $i = 1$
- $f(v_i) = 4$ for $i = 2$
- $f(v_i) = 1$ for $i = 1$
- $f(v_i) = i + n + 8$ for $2 \leq i \leq m + 1$
- $f(v_i) = 6$ for $i = m + 2$
- $f(v_i) = 7$ for $i = m + 3$
- $f(v_i) = 4$ for $i = m + 4$
- $f(v_i) = m + n + 10$ for $i = m + 5$
- $f(v_i) = n + 9$ for $i = m + 6$
- $f(v_i) = 8$ for $i = m + 7$
- $f(v_i) = 3$ for $i = m + 8$
- $f(v_i) = 5$ for $i = m + 9$
- $f(v_i) = 0$ for $i = m + 10$
- $f(v_i) = 2$ for $i = m + 11$
- $f(v_i) = i - m - 3$ for $m + 12 \leq i \leq m + n + 11$
Then, the labeling \( f \) on the vertices of \( RE_{m,5} * S_n \) of category II, type I, satisfies the conditions for a graceful labeling.

Hence complete the proof of theorem -2 e.

**The proof of Theorem -2:**

First, the graph \( RE_{m,q} \), it is defined as having center node, \( q \) legs of length 2 and \( m \) legs of length 1.

In this graph there are three classes of edges and four classes of vertices.

These will be referred to as the classes of outer, inner and single edges and center, middle, single and outer vertices respectively.

A leg of length one has a single edge which is incident with the center vertex and a single vertex.

The inner edge is incident with the center vertex and a middle vertex and the outer edge is incident with the middle vertex and an outer vertex.

A star graph \( S_n \), it has a center node and \( n \) legs of length one.

Then, the center node of the star \( S_n \) is attached with an outer vertex of the graph \( RE_{m,q} \) to form \( RE_{m,q} * S_n \). This is a tree with vertices \((m + 2q + n + 1)\) and edges \((m + 2q + n)\).

Secondly, let \( v_1, v_2, v_3, \ldots, v_{m+2q+1} \) are the vertices of the spider graph \( RE_{m,q} \) with the label \( v_1 \) represents the center and \( v_2, v_3, \ldots, v_{m+2q+1} \) are representing the single vertices and \( v_{m+2}, v_{m+3}, v_{m+4}, \ldots, v_{m+q+1} \), denoting the middle vertices and \( v_{m+q+2}, v_{m+q+3}, \ldots, v_{m+2q+1} \) indicating the outer vertices:

Thirdly, the center of the star \( S_n \) is affixed with the outer vertex \( v_{m+2q+1} \) and the other vertices of are labeled, by giving the labels \( v_{m+2q+2}, v_{m+2q+3}, \ldots, v_{m+2q+n+1} \) continuously starting from one pendent vertex till the last pendant vertex. The graph of \( RE_{m,q} * S_n \) as shown in the following figure - 9:

The labeling on the vertices of \( RE_{m,q} * S_n \) of category II, type I is defined by

\[
\begin{align*}
 f(v_i) &= 1 & \text{for } i = 1 \\
 f(v_i) &= 2q + n - 2 + i & \text{for } 2 \leq i \leq m + 1 \\
 f(v_i) &= 2i - 2m - 1 & \text{for } m + 2 \leq i \leq m + q - 1 \\
 f(v_i) &= i + q + n & \text{for } i = m + q \\
 f(v_i) &= i + q + n - m - 2 & \text{for } i = m + q + 1 \\
 f(v_i) &= 2m + 4q - 2i + 2 & \text{for } m + q + 2 \leq i \leq m + 2q - 1 
\end{align*}
\]
\[ f(v_i) = i - m - 2q \quad \text{for } i = m + 2q \]
\[ f(v_i) = i - m - 2q + 1 \quad \text{for } i = m + 2q + 1 \]
\[ f(v_i) = i - m - 3 \quad \text{for } m + 2q + 2 \leq i \leq m + 2q + n + 1 \]

Then, the labeling \( f \) on the vertices of \( RE_{m,q} \ast S_n \) of category II, type I, satisfies the conditions for a graceful labeling.

**Corollary - 2 - 1:**

The graph \( S_{q,2} \ast S_n \) of category II, type I is graceful for \( q \geq 1 \) and \( n \geq 1 \)

**Proof:**

In theorem -2, by putting \( m = 0 \), the graph \( S_{q,2} \ast S_n \) of category II, type I is constructed.

**Case - 1: \( q = 1 \)**

The labeling \( f \) on the vertices \( S_{1,2} \ast S_n \) of category II, type I it is defined by

\[ f(v_1) = 1 \quad \text{for } i = 1 \]
\[ f(v_i) = 1 + n \quad \text{for } i = 2 \]
\[ f(v_i) = i - 2 \quad \text{for } 4 \leq i \leq n + 3 \]

Then, the labeling \( f \) on the vertices of \( S_{1,2} \ast S_n \) of category II, type I satisfies the conditions for a graceful labeling.

**Case -2: \( q > 1 \)**

Put \( m = 0 \), in the labeling \( f \) defined in theorem -2 to get a graceful labeling of the graph \( S_{q,2} \ast S_n \) of category II, type I.
Figure 9. Ordinary labeling of $RE_{m,q} * S_n$ of category II, type I

Corollary - 2 - 2:

The graph $RE_{m,q}$, it is graceful for $q \geq 1$ and $m \geq 1$

Proof:

Put $n = 0$, in the theorem - 2 to construct the graph $RE_{m,q}$

**Case - 1: $q = 1$**

The labeling $f$ on the vertices of $RE_{m,1}$, it is defined by

\[
\begin{align*}
    f(v_i) &= i & \text{for } 1 \leq i \leq m + 2 \\
    f(v_i) &= i - m - 3 & \text{for } i = m + 3
\end{align*}
\]

Then, the labeling $f$ on the vertices of $RE_{m,1}$ satisfies the conditions for a graceful labeling.

**Case - 2: $q = 2$**

The labeling $f$ on the vertices of $RE_{m,2}$, it is defined by

\[
\begin{align*}
    f(v_i) &= i & \text{for } i = 1 \\
    f(v_i) &= i + 2 & \text{for } 2 \leq i \leq m + 2 \\
    f(v_i) &= i - m & \text{for } i = m + 3 \\
    f(v_i) &= i - m - 4 & \text{for } i = m + 4 \\
    f(v_i) &= i - m - 3 & \text{for } i = m + 5
\end{align*}
\]
Then, the labeling $f$ on the vertices of $RE_{m,1}$ satisfies the conditions for a graceful labeling.

**Case - 3: $q > 2$**

Put $n = 0$, in the labeling $f$ defined in theorem -2 to get a graceful labeling of the graph $RE_{m,q}$

**REFERENCES**


